Assignment 1 (T)

1. (Problem 1.1 from the textbook)
Show that if \( w \) is continuous on \([0, 1]\) and
\[
\int_0^1 w(x)v(x) \, dx = 0, \quad \forall v \in V,
\]
then \( w(x) = 0 \) for \( x \in [0, 1] \). Here
\[
V = \{ v : v \in C([0, 1]), \, \text{\( v' \) is piecewise continuous and bounded on } [0, 1], \, \text{and } v(0) = v(1) = 0 \}.
\]

2. (Problem 1.3 from the textbook)
Construct a finite-dimensional subspace \( V_h \) of \( V \) consisting of functions which are quadratic on each subinterval \( I_j \) of a partition of \( I = (0, 1) \). How can one choose the parameters to describe such functions? Find the corresponding basis functions. Then formulate a finite element method for the problem
\[
-u''(x) = f(x), \quad 0 < x < 1
\]
\[
u(0) = u'(0) = u(1) = u'(1) = 0.
\]
using the space \( V_h \) and write down the corresponding linear system of equations in case of a uniform partition.

3. (Problem 1.5 from the textbook)
Consider the boundary value problem
\[
\frac{d^4u}{dx^4} = f, \quad 0 < x < 1
\]
\[
u(0) = u'(0) = u(1) = u'(1) = 0.
\]
Here \( u \) represents e.g. the deflection of a clamped beam subject to a transversal force with intensity \( f \).
(a) Show that (1) can be given the following variational formulation: Find \( u \in W \) such that
\[
(u'', v'') = (f, v) \quad \forall v \in W,
\]
where \( W = \{ v : v \text{ and } v' \text{ are continuous on } [0, 1], \, v'' \text{ is piecewise continuous and } v(0) = v'(0) = v(1) = v'(1) = 0 \} \).
(b) For \( I = [a, b] \) an interval, define
\[
\mathbb{P}_3(I) = \{ v : v \text{ is a polynomial of degree } \leq 3 \text{ on } I, \, i.e., \, v \text{ has the form} \}
\[
v(x) = a_3x^3 + a_2x^2 + a_1x + a_0, \, x \in I \text{ where } a_i \in \mathbb{R} \}.
\]
Show that \( v \in \mathbb{P}_3 \) is uniquely determined by the values \( v(a), \, v'(a), \, v(b), \, v'(b) \). Find the corresponding basis functions (the basis function corresponding to the value \( v(a) \) is the cubic polynomial \( v \) such that \( v(a) = 1, \, v'(a) = v(b) = v'(b) = 0 \), etc).
(c) Starting from (b) construct a finite-dimensional subspace \( W_h \) of \( W \) consisting of piecewise cubic functions. Specify suitable parameters to describe the functions in \( W_h \) and determine the corresponding basis functions.
(d) Formulate a finite element method for (1) based on the space \( W_h \). Find the corresponding linear system of equations in the case of a uniform partition. Determine the solution in e.g. the case of two intervals and \( f \) constant. Compare with the exact solution.