Goals

- Introduce ordinary differential equations (ODEs) and initial value problems (IVPs).
- Examples of IVPs.
- Existence and uniqueness of solutions of IVPs.
- Dependence of the solution of an IVPs on parameters.
Given functions $f_1, \ldots, f_n : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ and scalars $y_{0,1}, \ldots, y_{0,n} \in \mathbb{R}$, we want to find $y_1, y_2, \ldots, y_n : \mathbb{R} \rightarrow \mathbb{R}$ such that

\[ y'_1(x) = f_1(x, y_1(x), \ldots, y_n(x)), \]
\[ y'_2(x) = f_2(x, y_1(x), \ldots, y_n(x)), \]
\[ \vdots \]
\[ y'_n(x) = f_n(x, y_1(x), \ldots, y_n(x)) \]  

(1) is called a system of first order ordinary differential equations (ODEs) and (2) are called initial conditions. Together it is called an initial value problem (IVP).
If we define

\[ f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n, \quad f(x, y_1, \ldots, y_n) = \begin{pmatrix} f_1(x, y_1 \ldots, y_n) \\ \vdots \\ f_n(x, y_1 \ldots, y_n) \end{pmatrix}, \]

\[ y : \mathbb{R} \rightarrow \mathbb{R}^n, \quad y(x) = \begin{pmatrix} y_1(x) \\ \vdots \\ y_n(x) \end{pmatrix}, \]

we can rewrite the system as

\[ y'(x) = f(x, y(x)) \]
\[ y(a) = y_0, \]

where \( y_0 = (y_{0,1}, \ldots, y_{0,n})^T \).
Often, one has to solve $n$-th order differential equations of the form

$$z^{(n)}(x) = g(x, z(x), z'(x), \ldots, z^{(n-1)}(x)), \quad x \in [a, b]$$

with initial conditions

$$z(a) = z_0, \quad z'(a) = z_1, \ldots, \quad z^{(n-1)}(a) = z_{n-1}.$$ 

(3) can be reformulated as a system of first order ODEs.
If we introduce the functions

\[
y_1(x) = z(x), \\
y_2(x) = z'(x), \\
\vdots \\
y_n(x) = z^{(n-1)}(x),
\]

then these functions satisfy the first order differential equations

\[
y_1'(x) = y_2(x), \\
y_2'(x) = y_3(x), \\
\vdots \\
y_{n-1}'(x) = y_n(x), \\
y_n'(x) = g(x, y_1(x), y_2(x), \ldots, y_n(x)),
\]

with initial conditions \( y_1(a) = z_0, y_2(a) = z_1, \ldots, y_n(a) = z_{n-1}. \)
$n$-th Order Differential Equation.

If we introduce

$$f(x, y_1, \ldots, y_n) = \begin{pmatrix} y_2(x) \\ y_3(x) \\ \vdots \\ y_n(x) \\ g(x, y_1(x), \ldots, y_n(x)) \end{pmatrix}$$

and

$$y_0 = \begin{pmatrix} z_0 \\ z_1 \\ \vdots \\ z_{n-1} \end{pmatrix},$$

then we arrive at the system of first order ODEs

$$y'(x) = f(x, y(x)), \quad y(a) = y_0.$$

Thus, it is sufficient to consider systems of first order ODEs.
Example 1 (Predator-Prey Model).

Let’s look at a special case of an interaction between two species, one of which the predators eats the other the prey.

Canadian lynx and snowshoe hare

Pictures and data are from http://www.math.duke.edu/education/ccp/materials/diffeq/predprey/pred1.html
Example 1 (Predator-Prey Model).

Mathematical model. Assumptions.

- The predator species is totally dependent on the prey species as its only food supply. $y(t)$ is the size of the predator population at time $t$.
- The prey species has an unlimited food supply; only predator poses threat to its growth. $x(t)$ is the size of the prey population at $t$. 
Example 1 (Predator-Prey Model).

Prey population.

- If there were no predators, the second assumption would imply that the prey species grows exponentially, i.e., \( x'(t) = ax(t) \).
- Since there are predators, we must account for a negative component in the prey growth rate. Assumptions:
  - The rate at which predators encounter prey is jointly proportional to the sizes of the two populations.
  - A fixed proportion of encounters leads to the death of the prey.

\[
x'(t) = ax(t) - bx(t)y(t).
\]

Predator population.

\[
y'(t) = -cy(t) + px(t)y(t).
\]
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\]

Predator population.

\[
y'(t) = -cy(t) + px(t)y(t).
\]

- **Lotka-Volterra Predator-Prey Model:** \((a, b, c, p > 0\) are constants)

\[
x'(t) = ax(t) - bx(t)y(t),
\]
\[
y'(t) = -cy(t) + px(t)y(t).
\]
Example 1 (Predator-Prey Model).

Solution of the Lotka-Volterra Predator-Prey Model

\[ x'(t) = ax(t) - bx(t)y(t), \]
\[ y'(t) = -cy(t) + px(t)y(t) \]

with initial condition

\[ x(0) = 20, \quad y(0) = 40 \]

and parameters

\[ a = 1, \quad b = 0.02, \quad c = 0.3, \quad p = 0.005. \]
Example 2 (Chemical Reactions).

- A reaction involving the compounds \( A, B, C, D \) is written as

\[
\sigma_A A + \sigma_B B \rightarrow \sigma_C C + \sigma_D D.
\]

Here \( \sigma_A, \sigma_B, \sigma_C, \sigma_D \) are the stoichiometric coefficients. The compounds \( A, B \) are the reactants, \( C, D \) are the products. The \( \rightarrow \) indicates that the reaction is irreversible. For a reversible reaction we use \( \rightleftharpoons \).

- For example, the reversible reaction of carbon dioxide and hydrogen to form methane plus water is

\[
CO_2 + 4H_2 \rightleftharpoons CH_4 + 2H_2O.
\]

The stoichiometric coefficients are

\[
\sigma_{CO_2} = 1, \, \sigma_{H_2} = 4, \, \sigma_{CH_4} = 1, \, \sigma_{H_2O} = 2.
\]

- For each reaction we have a rate \( r \) (the number of reactive events per second per unit volume, measured in \([\text{mol}/(\text{sec L})]\)) of the reaction that together with the stoichiometric coefficients determines the change in concentrations (measured in \([\text{mol}/\text{L}]\)) resulting from the reaction.
Example 2 (Chemical Reactions).

- For example, if the rate of the reaction is \( r \) and if denote the concentration of compound \( A, \ldots \) by \( C_A, \ldots \), we have the following changes in concentrations:

\[
\frac{d}{dt} C_A(t) = \cdots - \sigma_A r \cdots, \\
\frac{d}{dt} C_B(t) = \cdots - \sigma_B r \cdots, \\
\frac{d}{dt} C_C(t) = \cdots \sigma_C r \cdots, \\
\frac{d}{dt} C_D(t) = \cdots \sigma_D r \cdots,
\]

- Usually the reaction \( r \) is of the form

\[
r = k C_A^{\alpha} C_B^{\beta},
\]

where \( k \) is the reaction rate constant and \( \alpha, \beta \) are nonnegative parameters. The sum \( \alpha + \beta \) is called the order of the reaction.
Example 2 (Chemical Reactions. Autocatalytic reaction.)

- Autocatalysis is a term commonly used to describe the experimentally observable phenomenon of a homogeneous chemical reaction which shows a marked increase in rate in time, reaches its peak at about 50 percent conversion, and the drops off. The temperature has to remain constant and all ingredients must be mixed at the start for proper observation.

- We consider the catalytic thermal decomposition of a single compound $A$ into two products $B$ and $C$, of which $B$ is the autocatalytic agent. $A$ can decompose via two routes, a slow uncatalyzed one ($r_1$) and another catalyzed by $B$ ($r_3$). The three essential kinetic steps are

\[
A \rightarrow B + C \quad \text{Start or background reaction},
\]

\[
A + B \rightarrow AB \quad \text{Complex formation},
\]

\[
AB \rightarrow 2B + C \quad \text{Autocatalytic step}.
\]

- Again, we denote the concentration of compound $A, \ldots$ by $C_A, \ldots$. The reaction rates for the three reactions are

\[
r_1 = k_1 C_A, \quad r_2 = k_2 C_A C_B, \quad r_3 = k_3 C_{AB}.
\]
Example 2 (Chemical Reactions.)

The autocatalytic reaction leads to a system of ODEs

\[
\begin{align*}
\frac{dC_A}{dt} &= -k_1 C_A - k_2 C_A C_B, \\
\frac{dC_B}{dt} &= k_1 C_A - k_2 C_A C_B + 2k_3 C_{AB}, \\
\frac{dC_{AB}}{dt} &= k_2 C_A C_B - k_3 C_{AB}, \\
\frac{dC_C}{dt} &= k_1 C_A + k_3 C_{AB}
\end{align*}
\]

with \( k_1 = 0.0001,\ k_2 = 1,\ k_3 = 0.0008 \) and initial values \( C_A(0) = 1,\ C_B(0) = 0,\ C_{AB}(0) = 0,\ C_C(0) = 0 \).

Here time is measured in \([\text{sec}]\) and concentrations are measured in \([\text{kmol/L}]\).
Example 3.

For given $A \in \mathbb{R}^{n \times n}$ consider the linear IVP

$$y'(x) = Ay(x), \quad x \in [0, \infty)$$
$$y(0) = y_0. \quad (4)$$

Suppose there exist $V = (v_1 | \ldots | v_n) \in \mathbb{C}^{n \times n}$ and

$\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n), \ \lambda_j \in \mathbb{C}$, such that (eigen decomposition)

$$A = V \Lambda V^{-1}$$

or, equivalently,

$$A v_j = \lambda_j v_j, \quad j = 1, \ldots, n.$$

Insert into (4)

$$(V^{-1}y)'(x) = V^{-1}y'(x) = V^{-1}AVV^{-1}y(x) = \Lambda V^{-1}y(x), \quad x \in [0, \infty)$$
$$V^{-1}y(0) = V^{-1}y_0.$$
Example 3.

- Putting \( z := V^{-1} y \) we can see that \( z \) satisfies

\[
  z'(x) = \Lambda z(x), \quad x \in [0, \infty),
\]
\[
  z(0) = V^{-1} y_0 := z_0.
\]

- Since \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n) \), this is equivalent to

\[
  z'_j(x) = \lambda_j z_j(x), \quad x \in [0, \infty),
\]
\[
  z_j(0) = z_{0,j}, \quad j = 1, \ldots, n.
\]

- Unique solution

\[
  z_j(x) = e^{\lambda_j x} z_{0,j}, \quad j = 1, \ldots, n.
\]

- Hence, the unique solution of (4) is given by

\[
  y(x) = V z(x) = \sum_{j=1}^{n} v_j e^{\lambda_j x} z_{0,j}.
\]
Consider

\[
A = \begin{pmatrix}
-20.08 & -39.96 \\
-39.96 & -80.02
\end{pmatrix}
\]

We can show

\[
A = \frac{1}{\sqrt{5}} \begin{pmatrix}
1 & -2 \\
2 & 1
\end{pmatrix} \begin{pmatrix}
-100 & 0 \\
0 & -0.1
\end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix}
1 & 2 \\
-2 & 1
\end{pmatrix} = V \Lambda V^{-1}.
\]

and \( y_0 = (1, 1)^T \). Since

\[
V^{-1}y_0 = \frac{1}{\sqrt{5}} \begin{pmatrix}
1 & 2 \\
-2 & 1
\end{pmatrix} \begin{pmatrix}
1 \\
1
\end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix}
3 \\
-1
\end{pmatrix} := z_0
\]
Example 3.

This leads to

\[ z_1'(x) = -100z_1(x), \quad z_1(0) = \frac{3}{\sqrt{5}} \]

\[ z_2'(x) = -0.1z_2(x), \quad z_2(0) = -\frac{1}{\sqrt{5}} \]
Existence and Uniqueness of Solutions.

Theorem (Existence Theorem of Peano)

Let $D \subset \mathbb{R}^{n+1}$ be a domain, i.e. an open connected subset of $\mathbb{R}^n$ with $(a, y_0) \in D$. If $f$ is continuous on $D$, then there exists $\delta > 0$ such that the IVP

$$y'(x) = f(x, y(x)), \quad y(a) = y_0.$$ 

has a solution on the interval $[a - \delta, a + \delta]$. 
Example
Consider the IVP

\[ y'(x) = y(x)^{1/3}, \quad y(0) = 0, \quad x \geq 0. \]

For arbitrary \( \bar{x} > 0 \) the functions

\[ y(x) = \begin{cases} 
0, & 0 \leq x \leq \bar{x} \\
\pm \frac{2}{3} [(x - \bar{x})]^{3/2}, & x \geq \bar{x}
\end{cases} \]

solve the IVP. Hence, the IVP has infinitely many solutions. Note that the partial derivative of \( f(x, y) = y^{1/3} \) with respect to \( y \) is singular at \( y = 0 \).
Existence and Uniqueness of Solutions.

Theorem (Existence and Uniqueness Theorem of PicardLindelöf)

If $f$ is continuous on $D$ and if there exists $M > 0$ such that
$\|f(x,y)\| \leq M$ for all $(x,y) \in D$ and if $f$ is Lipschitz continuous with
respect to $y$ on

$$\tilde{D} = \{(x,y) \in D : |x-a| \leq \delta, \|y-y_0\| \leq \delta M\},$$

i.e., if there exists $L > 0$ such that

$$\|f(x,y_1) - f(x,y_2)\| \leq L\|y_1 - y_2\| \quad \text{for all} \quad (x,y_1),(x,y_2) \in \tilde{D},$$

then

$$y'(x) = f(x,y(x)),$$

$$y(a) = y_0.$$

has a unique solution on the interval $[a - \delta, a + \delta]$. 
Existence and Uniqueness of Solutions.

The Picard Lindelöf Theorem is based on the equivalence of the IVP

\[ y'(x) = f(x, y(x)), \quad x \in [a, b], \]
\[ y(a) = y_0. \]

and the integral equation

\[ y(x) = y_0 + \int_a^x f(s, y(s))ds. \]

The integral of a vector valued function is defined component wise, i.e.,

\[ \int_a^x f(s, y(s))ds = \begin{pmatrix} \int_a^x f_1(s, y(s))ds \\ \vdots \\ \int_a^x f_n(s, y(s))ds \end{pmatrix}. \]
Consider the two ODEs

\[ y'(x) = f(x, y(x)), \]
\[ y(a) = y_0 \] \hspace{1cm} (5)

and

\[ z'(x) = g(x, z(x)), \]
\[ z(a) = z_0, \] \hspace{1cm} (6)

where \( f, g : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n \) are given functions and \( y_0, z_0 \in \mathbb{R}^n \) are given vectors.

We view (6) as a perturbation of (5) and we want to know what the error between the exact solution \( y \) and the solution of the perturbed problem \( z \) is.

This question is also interesting in the context of numerical solutions of IVPs.
Dependence of the Solution of the ODE on Perturbations of the Problem.

- Suppose there exist constants $\epsilon_1, \epsilon_2 > 0$ such that
  \[ ||y_0 - z_0|| \leq \epsilon_1 \]
  and
  \[ ||f(x, y) - g(x, y)|| \leq \epsilon_2 \quad \forall x \in \mathbb{R}, y \in \mathbb{R}^n. \]

Furthermore, suppose there exists $L > 0$ such that
\[ ||f(x, y) - f(x, z)|| \leq L ||y - z|| \quad \forall x \in \mathbb{R}, y, z \in \mathbb{R}^n. \]

- Suppose that the IVPs (5) and (6) have unique solutions $y$ and $z$.
- These solutions satisfy the integral equations
  \[ y(x) = y_0 + \int_a^x f(s, y(s))ds \quad (7) \]
  and
  \[ z(x) = z_0 + \int_a^x g(s, z(s))ds. \quad (8) \]
Dependence of the Solution of the ODE on Perturbations of the Problem.

Subtract (8) from (7) to get

\[ y(x) - z(x) = y_0 - z_0 + \int_a^x f(s, y(s)) - g(s, z(s)) \, ds \]

\[ = y_0 - z_0 + \int_a^x f(s, y(s)) - f(s, z(s)) \, ds + \int_a^x f(s, z(s)) - g(s, z(s)) \, ds. \]
Dependence of the Solution of the ODE on Perturbations of the Problem.

Taking the norm we get

\[ \|y(x) - z(x)\| \leq \|y_0 - z_0\| + \left\| \int_a^x f(s, y(s)) - f(s, z(s)) \, ds \right\| \]

\[ + \left\| \int_a^x f(s, z(s)) - g(s, z(s)) \, ds \right\| \]

\[ \leq \|y_0 - z_0\| + \int_a^x \left( \|f(s, y(s)) - f(s, z(s))\| \right) \, ds \]

\[ \leq \epsilon_1 + \int_a^x \left( L \|y(s) - z(s)\| \right) \, ds \]

\[ + \int_a^x \left( \|f(s, z(s)) - g(s, z(s))\| \right) \, ds \leq \epsilon_2 \]
Dependence of the Solution of the ODE on Perturbations of the Problem.

Hence if we define the error

\[ e(x) = \| y(x) - z(x) \|, \]

then

\[ e(x) \leq \epsilon_1 + L \int_a^x e(s) ds + \epsilon_2 (x - a). \]
Lemma (Gronwall Lemma)

If $h, w$ and $k$ are nonnegative and continuous on the interval $[a, b]$ satisfying the inequality

$$h(x) \leq w(x) + \int_a^x k(s)h(s)ds \quad \forall x \in [a, b],$$

then $h$ obeys the estimate

$$h(x) \leq w(x) + \int_a^x e^{\int_t^x k(s)ds} k(t)w(t)dt.$$
Dependence of the Solution of the ODE on Perturbations of the Problem.

Apply the Gronwalls lemma to the equation

\[ e(x) \leq \epsilon_1 + \epsilon_2(x - a) + L \int_a^x e(s)ds. \]

Here \( h(x) = e(x) = \|y(x) - z(x)\| \), \( w(x) = \epsilon_1 + \epsilon_2(x - a) \), and \( k(x) = L \).

Thus, the error between the solution \( y \) of the original IVP (5) and the solution \( z \) of the perturbed IVP (6) obeys

\[
\|y(x) - z(x)\| \leq \epsilon_1 + \epsilon_2(x - a) + \int_a^x e^{L(x-t)}L[\epsilon_1 + \epsilon_2(t - a)]dt
\]

\[ = \epsilon_1 e^{L(x-a)} + \frac{\epsilon_2}{L}(e^{L(x-a)} - 1) \quad x \geq a. \]
Dependence of the Solution of the ODE on Perturbations of the Problem.

Example

- The solution of the differential equation

\[ y'(x) = 3y(x), \quad y(0) = 1 \]

is given by \( y(x) = e^{3x} \). The function \( f(x, y) = 3y \) is Lipschitz continuous with respect to \( y \) with Lipschitz constant \( L = 3 \).

- The perturbed differential equation

\[ z'(x) = 3z(x) + \epsilon_2, \quad z(0) = 1 + \epsilon_1 \]

has the solution \( z(x) = (1 + \epsilon_1 + \frac{\epsilon_2}{3})e^{3x} - \frac{\epsilon_2}{3} \).

- Thus,

\[ |y(x) - z(x)| = \epsilon_1 e^{3x} + \frac{\epsilon_2}{3} (e^{3x} - 1) \quad x \in [0, \infty) \]

This shows that the error estimate (9) is sharp.