Gauss Quadrature.
Composite Quadrature Formulas.
MATLAB’s Functions.
Gauss Quadrature.

- Idea of the Gauss Quadrature is to choose nodes $x_0, \ldots, x_n$ and the weights $w_0, \ldots, w_n$ such that the formula

$$
\int_a^b p(x) \, dx \approx \sum_{i=0}^n w_i p(x_i).
$$

is exact for a polynomial of maximum degree.

- **Lemma**

  There is no choice of nodes $x_0, \ldots, x_n$ and weights $w_0, \ldots, w_n$ such that

$$
\int_a^b p_N(x) \, dx \approx \sum_{i=0}^n w_i p_N(x_i).
$$

for all polynomials $p_N$ of degree less or equal to $N$ if $N > 2n + 1$.

- The above lemma give an upper bound on the maximum degree.
Gauss Quadrature.

Example
Let’s determine the weights $w_0$ and $w_1$ and the nodes $x_0$ and $x_1$ such that

$$w_0 p(x_0) + w_1 p(x_1) = \int_{-1}^{1} p(x) \, dx$$

holds for polynomials of degree 3 or less.

This seems possible since we have 4 parameters to choose $w_0, w_1, x_0, x_1$ and exactly 4 numbers are needed in order to define uniquely a polynomial of degree 3.
Gauss Quadrature.

Example
Let’s force the formula to be exact for $1$, $x$, $x^2$, and $x^3$. This gives us

$$w_1 + w_2 = \int_{-1}^{1} 1 \, dx = 2$$

$$w_1 x_1 + w_2 x_2 = \int_{-1}^{1} x \, dx = 0$$

$$w_1 x_1^2 + w_2 x_2^2 = \int_{-1}^{1} x^2 \, dx = \frac{2}{3}$$

$$w_1 x_1^3 + w_2 x_2^3 = \int_{-1}^{1} x^3 \, dx = 0.$$ 

a nonlinear system of 4 equations with 4 unknowns.
Usually we need a nonlinear solver to solve nonlinear systems, but in this example we can solve it analytically to obtain

$$w_1 = w_2 = 1, \quad x_1 = -\frac{1}{\sqrt{3}}, \quad x_2 = \frac{1}{\sqrt{3}}.$$
The weights and nodes for the first 3 Gauss-Legendre formulas on $[-1, 1]$.

<table>
<thead>
<tr>
<th>$x_i$</th>
<th>$w_i$</th>
<th>exact for $p_N$,</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$</td>
<td>1,1</td>
<td>$N = 3$</td>
</tr>
<tr>
<td>$-\sqrt{\frac{3}{5}}, 0, \sqrt{\frac{3}{5}}$</td>
<td>$\frac{5}{9}, \frac{8}{9}, \frac{5}{9}$</td>
<td>$N = 5$</td>
</tr>
<tr>
<td>-0.861136311594052575</td>
<td>0.34785484513745385737</td>
<td>$N = 7$</td>
</tr>
<tr>
<td>-0.339981043584856264</td>
<td>0.65214515486254614262</td>
<td></td>
</tr>
<tr>
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</table>
Composite Quadrature Formulas.

Let $a = x_0 < x_1 < \cdots < x_n = b$ be a partition of $[a, b]$ Then

$$
\int_a^b f(x) \, dx = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(x) \, dx.
$$

Now we can approximate $\int_a^b f(x) \, dx$ by approximating each integral $\int_{x_i}^{x_{i+1}} f(x) \, dx$ by a (low degree) quadrature formula,

$$
\int_{x_i}^{x_{i+1}} f(x) \, dx \approx \sum_{j=0}^{m} w_{ji} f(x_{ji})
$$

and

$$
\int_a^b f(x) \, dx = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(x) \, dx \approx \sum_{i=0}^{n-1} \sum_{j=0}^{m} w_{ji} f(x_{ji}).
$$
Composite Midpoint Rule.

Example

\[
\int_a^b f(x) \, dx \approx \sum_{i=0}^{n-1} (x_{i+1} - x_i) f \left( \frac{x_{i+1} + x_i}{2} \right)
\]
Composite Trapezoidal Rule.

Example

\[
\int_a^b f(x) \, dx \approx \sum_{i=0}^{n-1} \frac{x_{i+1} - x_i}{2} (f(x_{i+1}) + f(x_i))
\]

The function values \(f(x_1), f(x_2), \ldots, f(x_{n-1})\) appear twice in the summation. This has to be utilized in the implementation of the composite Trapezoidal rule:

\[
\int_a^b f(x) \, dx \approx \frac{x_1 - x_0}{2} f(x_0) + \sum_{i=1}^{n-1} \left( \frac{x_i - x_{i-1}}{2} + \frac{x_{i+1} - x_i}{2} \right) f(x_i) + \frac{x_n - x_{n-1}}{2} f(x_n)
\]
Composite Simpsons Rule.

Example

\[ \int_{a}^{b} f(x) \, dx \approx \frac{n-1}{6} \sum_{i=0}^{n-1} \left( x_{i+1} - x_i \right) \left( f(x_i) + 4f\left( \frac{x_{i+1} + x_i}{2} \right) + f(x_{i+1}) \right). \]

Notice that the function values \( f(x_1), f(x_2), \ldots, f(x_{n-1}) \) appear twice in the summation. This has to be utilized in the implementation of the composite Simpson rule.
MATLAB has several built-in functions for numerical integration. We will mention a couple: `quad` and `trapz`. You can get more information by typing

```matlab
>> help quad
>> help trapz
```
MATLAB’s quad

The syntax for quad

QUAD Numerically evaluate integral, adaptive Simpson quadrature.

Q = QUAD(FUN,A,B) tries to approximate the integral of scalar-valued function FUN from A to B to within an error of 1.e-6 using recursive adaptive Simpson quadrature.

FUN is a function handle.

The function Y = FUN(X) should accept a vector argument X and return a vector result Y, the integrand evaluated at each element of X.
Example.

For example we want to approximate

\[ \int_{0}^{10} e^{-x^2} \, dx. \]

Then

>> quad('exp(-x.^2)',0,10)

produces

>> 0.886226046613606

If we need 10 digits of accuracy then

>> quad('exp(-x.^2)',0,10,1e-10)

produces more accurate answer

>> 0.886226925457492