STU's PUZZLE CORNER
by Stuart Sidney

Powers of sums and sums of powers

Way back when you first learned about the notion of *definite integral*, you may have evaluated a few directly from the definition. Typically, the first few integrals were of monomials x^a with a a positive integer, say 1, 2 or 3, and to actually evaluate such an integral you needed to find a formula for $S_a(n) = \sum\{k^a : k = 1, \ldots, n\}$. If you did the cases $a = 1$ and $a = 3$, you couldn't help noticing that $(S_1(n))^2 = S_3(n)$ is true for every positive integer n. Perhaps you wondered whether there were any other equalities of this type. Precisely, always assuming that a, b and c are positive numbers and $b > 1$, are there any triples (a, b, c) for which

$$(*) \quad (S_a(n))^b = S_c(n) \qquad \text{for all } n$$

other than the triple $(a, b, c) = (1, 2, 3)$? Our three problems study this question in increasing generality. The first is well known, and not too hard; the second is somewhat harder; and as for the third, as of this writing, your diligent columnist has yet to settle the matter.

Problem 1. Prove that if a, b and c are positive *integers* with $b > 1$, then (*) holds only if $(a, b, c) = (1, 2, 3)$.

Problem 2. Prove that if a, b and c are positive *rational numbers* with $b > 1$, then (*) holds only if $(a, b, c) = (1, 2, 3)$.

Problem 3. Determine whether it is true that the only triple of positive *real numbers* (a, b, c) with $b > 1$ for which (*) holds is $(a, b, c) = (1, 2, 3)$.

Alas, **Problem 3** is not a famous problem with a big cash reward attached to it (recall the $100,000 Beal conjecture from last year's column), so the best I can offer is publication of the first correct solution. Along those lines, last year's puzzle column *did* elicit some response. I invited solutions to the following problem:

2003 Problem 2. Do the equations $a^2 + b^2 = c^4$, $a^4 + b^4 = c^2$, and $a^2 + b^3 = c^4$ have solutions in natural numbers?

Tom Leibowitz provided the solution $7^2 + 24^2 = 5^4$ to the first equation, which is a cousin to my solution $15^2 + 20^2 = 5^4$. Mine arose by starting with $3^2 + 4^2 = 5^2$ and deducing $(3\cdot5)^2 + (4\cdot5)^2 = (5\cdot5)^2$; more generally, $x^2 + y^2 = z^2$ leads to $(x\cdot z)^2 + (y\cdot z)^2 = (z\cdot z)^2$.
The second equation has *no solution* (in natural numbers), while one solution to the third is given by $6000^2 + 400^3 = 100^4$. To arrive at this last equation, start with $6^2 + 4^3 = 10^2$ and look for natural number exponents k and l for which $(6\cdot10^k)^2 + (4\cdot10^l)^3 = 10^m$ where m is a multiple of 4; thus $m = 2k+2 = 3l+2$, and the choice $k = 3$, $l = 2$ does the job.

Let's keep alumni input pouring (or at least trickling) in. Please send me your suggestions or solutions, by e-mail to: sidney@math.uconn.edu or *via* surface mail to: Stuart Sidney,
Department of Mathematics, Unit 3009, The University of Connecticut, Storrs, CT 06269-3009.