Rank functions and quiver representations

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Main points

- The rank of a linear map can be naturally generalized (in many ways) to quiver representations.
- The structure of the category of quivers over Q is useful for studying representations of Q.

- **Quiver** Q: finite directed graph.
- **Representation** $V \in \mathrm{Rep}(Q)$: collection of vector spaces and linear maps:

$$\{V_x \mid x \text{ vertex of } Q\} \qquad \{V_a \colon V_{ta} \to V_{ha} \mid a \text{ arrow of } Q\}.$$

Example:

$$Q = \begin{array}{ccccc} 1 & \overset{a}{\searrow} & & & \\ & & 3 & \xrightarrow{c} & 4 \\ 2 & \underset{b}{\nearrow} & & & \end{array} \qquad V = \begin{array}{ccccc} V_1 & \overset{V_a}{\searrow} & & & \\ & & V_3 & \xrightarrow{V_c} & V_4 \\ V_2 & \underset{V_b}{\nearrow} & & & \end{array}$$

- Appropriate morphisms make $\mathrm{Rep}(Q)$ into an abelian category.

Quiver tensor product

Definition

Tensor product in Rep(Q) is taken “pointwise":

$$V \otimes W = \begin{array}{c} V_1 \otimes W_1 \xrightarrow{V_a \otimes W_a} \\ \\ V_2 \otimes W_2 \xrightarrow{V_b \otimes W_b} \end{array} V_3 \otimes W_3 \xrightarrow{V_c \otimes W_c} V_4 \otimes W_4$$

- functorial in V, W
- associative, commutative, and distributes over $\oplus$.

Natural question

- Rep(Q) has a **unique decomposition property**: any decomposition

$$V \simeq \bigoplus_i V_i, \qquad V_i \text{ indecomposable},$$

is the same up to reindexing.

- Question: Given $V, W \in \text{Rep}(Q)$, what are the indecomposable $\oplus$-ands of $V \otimes W$?
- Analogous quesiton studied for finite groups, Lie groups, Lie algebras.
- Difference: no good description of the indecomposable representations of most quivers, and no character theory for quivers. So where to start?

Representation ring of a quiver

Idea: build a ring from the representations of Q, in which addition is direct sum and multiplication is tensor product.

Definition

Let

$$R(Q) := \frac{\text{Free abelian group on } [V] \in \text{Rep}(Q)}{\{[V \oplus W] - [V] - [W]\}}$$

The operation

$$[V] \cdot [W] := [V \otimes W] \qquad \text{for } V,\ W \in \text{Rep}(Q)$$

makes $R(Q)$ into a commutative ring with identity, the **representation ring** of Q.

Rank of a quiver representation

Let Q be an arbitrary (connected) quiver. Define functors $\mathscr{E}, \mathscr{M}, \mathscr{R} \colon \operatorname{Rep}(Q) \to \operatorname{Rep}(Q)$ by:

- $\mathscr{E}(V) \hookrightarrow V$ maximal such that each $\mathscr{E}(V)_a$ surjective
- $V \twoheadrightarrow \mathscr{M}(V)$ maximal such that each $\mathscr{M}(V)_a$ injective
- $\mathscr{R}(V) =$ image of composition $\mathscr{E}(V) \hookrightarrow V \twoheadrightarrow \mathscr{M}(V)$.

Then each linear map $\mathscr{R}(V)_a$ is an isomorphism. Define the **rank** of V to be the dimension of $\mathscr{R}(V)$ at any vertex. Write $r_Q(V) \in \mathbb{N} = \{0, 1, 2, \dots\}$.

Bonus: definition valid for representations of Q in more general categories (e.g. coherent sheaves on a Noetherian scheme).

Examples

- For $V = \; V_0 \xrightarrow{f_1} V_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} V_n$, we have
$$r_Q(V) = \operatorname{rank} f_n \cdots f_2 f_1 .$$
- Evident here that r_Q is additive and multiplicative (w.r.t. $\oplus$ and $\otimes$). This holds for any quiver. Also respects duality, and commutes with (quiver) Schur functors in char 0.
- Generally not a “nice” description, for example

$$r_Q(\; V_1 \xrightarrow{A} V_2 \xleftarrow{B} V_3 \xrightarrow{C} V_4 \;) = \dim_K \left(\frac{\operatorname{Im} A \cap \operatorname{Im} B}{\operatorname{Im} A \cap B(\ker C)} \right)$$

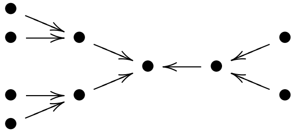
Generalized rank functions

- A map of directed graphs $\alpha\colon P \to Q$ (i.e. **quiver over** Q) induces a pullback functor $\alpha^*\colon \operatorname{Rep}(Q) \to \operatorname{Rep}(P)$ and then a **rank function** $r_\alpha = r_P \circ \alpha^*\colon \operatorname{Rep}(Q) \to \mathbb{N}$.
- Extending by linearity gives a ring homomorphism

$$r_\alpha\colon R(Q) \to \mathbb{Z}.$$

Rank functions on rooted tree quivers

- A **rooted tree quiver** is a tree with a unique sink.

Example: 

Fix rooted tree Q, and consider rooted trees over Q.

- Distinct α, β can have $r_\alpha = r_\beta$.
- Each r_α induced by minimal representative, $Q_\alpha \xrightarrow{\overline{\alpha}} Q$. Only need to understand these.

Combinatorics of rank functions

- Can show that $r_\alpha \geq r_\beta$ if and only if $\exists \gamma$ such that

$$\begin{array}{ccc} Q_\alpha & \xrightarrow{\gamma} & Q_\beta \\ & \searrow^{\overline{\alpha}} \quad {}^{\overline{\beta}}\swarrow & \\ & Q & \end{array} \qquad \text{commutes.}$$

- Let L_Q be the poset of rank functions on Q. Can show that L_Q is *finite*.
- There are **Galois connections** (a.k.a. **adjunctions**)

$$L_Q \underset{\pi_*}{\overset{\pi^*}{\rightleftarrows}} L_{Q_\alpha}$$

which can be used to relate $R(Q)$ and $R(Q_\alpha)$.

Induction process

- Why do we want to relate $R(Q)$ to the various $R(Q_\alpha)$? Since Q_α can have more vertices than Q, no obvious induction.
- Can totally order set of all rooted trees by **complexity** so that $Q_\alpha < Q$ unless $r_\alpha = r_Q$. Understand most of $R(Q)$ by induction this way.
- For remaining piece, use induction on values of r_α, and poset L_Q (i.e. minimal r_α which vanishes).

Theorem

If Q is a rooted tree quiver, the poset L_Q of rank functions on Q is finite. The rank functions give a ring isomorphism

$$R(Q)_{red} \xrightarrow{\sim} \prod_{M \in L_Q} \mathbb{Z}.$$

The idempotents on the right hand side lift to a complete set of orthogonal idempotents in $R(Q)$, which are explicitly described in terms of certain representations of Q.

Also get a **splitting principle** for $V^{\otimes n}$, $n \gg 0$.

Summary

- The rank of a linear map has natural generalizations to quiver representations.
- The structure of the category of quivers over Q is useful for studying representations of Q.

<u>Future directions</u>

- For which other quivers Q is $R(Q)_{red}$ module finite? Shown: Classified by finite set of forbidden minors.
- What are the geometric properties of global rank functions? Is r_Q constructible? What are **rank loci**?

For Further Reading

R. Kinser
The rank of a quiver representation,
J. Algebra 320(6):2363-2387, (2008)
arXiv:0711.1135

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Rank functions on rooted tree quivers,
arXiv:0807.4496