

GABRIEL'S THEOREM AND REFLECTION FUNCTORS

ABSTRACT. First, we'll review the argument for why only Dynkin quivers could possibly be of finite type, in order to set up some vocabulary and notation regarding root systems and quivers. Then we'll introduce reflection functors as a tool to show that all Dynkin quivers actually are of finite type. We'll make sure to see some examples of how to actually compute things.

The reflection functors appeared in a 1973 paper of Bernstein, Gel'fand, and Ponomarev [BGP73]. Time permitting, it would be nice to discuss the 2006 paper of Derksen, Weyman, and Zelevinsky [DWZ07] that generalizes these reflection functors and connects them with mutations in cluster algebras. This is a highly active area of current research.

1. ROOT SYSTEMS IN THE SPACE OF DIMENSION VECTORS

Working over algebraically closed field K for simplicity, &c... We're discussing

Theorem 1 (Gabriel). *A connected quiver Q is of finite representation type if and only if Q is Dynkin (of type ADE).*

First, to technically simplify things, we'll assume that none of the quivers we talk about today have cycles (oriented or not). The reason is that a quiver with cycles is not of ADE type, nor can it be possibly be of finite representation type. To see the last claim, just put 1 dimensional spaces K at all the vertices in a cycle, with the identity map at all arrows in the cycle except one, where we put an arbitrary scalar λ . The infinitely many different values of λ give infinitely many non-isomorphic representations. When we refer to Dynkin diagrams, we will always mean of ADE type, since there is no standard interpretation (or immediate relevance) of the other types in terms of quiver representations.

Now for any quiver (or graph), we have an \mathbb{R} -vector space E of "dimension vectors": just like we defined the dimension vector of a representation, but we'll allow arbitrary \mathbb{R} entries at each vertex. We write ε_x for the dimension vector of the simple representation supported at a vertex x . For example:

$$Q = \begin{array}{c} 1 \\ \searrow \\ \\ \nearrow \\ 2 \end{array} \longrightarrow 3 \longrightarrow 4 \qquad E = \left\{ \begin{array}{ccc} d_1 & & \\ & d_3 & d_4 \\ d_2 & & \end{array} \right\} \qquad \varepsilon_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Earlier, we looked at the symmetrized **Ringel form** $(\ , \)$ on E , which was defined in terms of Q (and only depended on the underlying graph). This bilinear form on E gives us **reflections**. We computed $s_x : E \rightarrow E$, the **simple reflection** in the hyperplane orthogonal to ε_x :

$$s_x(\varepsilon_y) = \varepsilon_y - 2(\varepsilon_y, \varepsilon_x)\varepsilon_x = \begin{cases} -\varepsilon_x & \text{if } x = y \\ \varepsilon_y + \varepsilon_x & \text{if } x, y \text{ connected by an edge} \\ \varepsilon_y & \text{otherwise} \end{cases}$$

(note how the assumption of no loops of multiple edges simplifies this computation). Since these reflections are linear, we can easily compute the reflection of any vector over our coordinate hyperplanes. Example:

$$\begin{matrix} 0 & 0 & 1 & \xrightarrow{s_3} & 0 & 1 & 1 & \xrightarrow{s_1} & 1 & 1 & 1 & \xrightarrow{s_2} & 1 & 1 & 1 & \xrightarrow{s_3} & 1 & 2 & 1 \\ 0 & 0 & 1 & & 0 & 1 & 1 & & 0 & 1 & 1 & & 1 & 1 & 1 & & 1 & 2 & 1 \end{matrix}$$

These reflections form a group (note that $s_x^{-1} = s_x$ by direct computation). From the magical world of “combinatorics”, we get

(super mega fact [Hum78]). *In the setup above, the following are equivalent:*

- (1) *The form $(\ , \)$ is positive definite*
- (2) *The group $W = \langle s_1, \dots, s_n \rangle$ generated by the simple reflections is finite*
- (3) *Q is Dynkin*

We saw before that if Q is of finite representation type, then $(\ , \)$ must be positive definite (by Tits’s geometric argument). Hence by the purely combinatorial mega fact above, we get Q must be Dynkin. So half of Gabriel’s theorem is proven; we just need to show that every Dynkin quiver has only finitely many indecomposable representations.

Hmmmm... since the reflection group W is finite, reflecting the simple roots ε_x by the s_y ’s will only generate finitely many more vectors, (called **roots**). Maybe there is some connection...

But wait: so far we have only used combinatorial and geometric arguments involving dimension vectors—we haven’t actually written down any representations in our setup. So even if we knew that indecomposable representations had something to do with roots, a priori we have no way of knowing that there aren’t infinitely many non-isomorphic representations having the same dimension vector. So we’ll need some stronger tools.

Before developing these tools, we’ll give a simpler, more direct proof of the half of Gabriel’s theorem that we’ve already done, which doesn’t invoke any combinatorial mega facts. Every subdiagram of a Dynkin diagram is Dynkin. Thus there are minimal non-Dynkin diagrams, and every non-Dynkin diagram must contain one as subdiagram. These are called **Euclidean** or **affine Dynkin** diagrams, of type \tilde{A} , \tilde{D} , and \tilde{E} . For any Euclidean quiver Q , we simply exhibit an infinite family of non-isomorphic representations of Q . Then this family extends by 0 to any quiver containing Q . We’ll indicate what these families are with a few examples: technically, they depend on the orientation of Q , but knowing the family for one orientation, one can easily modify it for other orientations.

$$\tilde{D}_4 : \begin{array}{c} 1 \\ \downarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow 2 \xleftarrow{\quad} 1 \\ \uparrow \begin{pmatrix} \mu \\ \lambda \end{pmatrix} \\ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ 1 \end{array} \quad \text{and for } \tilde{D}_n, \text{ add more 2-d spaces with identity maps between the branch points}$$

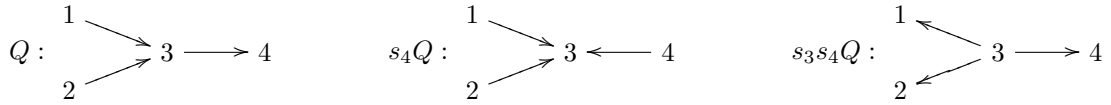
I don’t really feel like L^AT_EXing anything more complicated, so I’ll refer you to Ringel’s paper [Rin80]. He explicitly writes out the matrices for these minimal infinite families of representations. Euclidean quivers are not of finite type, but they are exactly the ones of “tame” representation type... but that’s another story.

Some philosophico-mathematical commentary: this gives some better view of how there is a threshold of complexity for representations of a quiver. Dynkin diagrams are below this threshold, everything else is

above it. It should be noted that the dimension vectors one gets from these “minimal infinite families” of representations are exactly the vectors that one would use in showing that the Cartan matrix of a non-Dynkin diagram is not positive definite.

2. REFLECTION FUNCTORS

So we can manipulate dimension vectors via “reflections at a vertex”, and the idea is to extend this be able to “reflect” representations at vertices. But we won’t be able to reflect a representation over an arbitrary vertex: we can only reflect at **sinks** and **sources**. Furthermore, when we reflect a representation at a sink or source x , we don’t get a representation of the same quiver, but rather of a **reflected quiver** $s_x Q$. This reflected quiver $s_x Q$ has the same vertices and edges as Q , but we reverse the orientation of all edges connected to x . E.g.



We use the same notation s_x for the reflection operations on quivers as for the reflections in the vector space E .

For a sink $x \in Q_0$, we’ll get a functor

$$C_x^+ : \text{Rep}(Q) \rightarrow \text{Rep}(s_x Q)$$

and if x is a source, we get a functor

$$C_x^- : \text{Rep}(Q) \rightarrow \text{Rep}(s_x Q)$$

When applied to a representation V , these functors only alter the space $V(x)$, and leave the other spaces unchanged. They can be summarized by

$$\begin{aligned}
 0 \rightarrow C_x^+ V(x) \rightarrow \bigoplus_{ha=x} V(ta) \rightarrow V(x) \\
 V(x) \rightarrow \bigoplus_{ta=x} V(ta) \rightarrow C_x^- V(x) \rightarrow 0
 \end{aligned}$$

which gives induced maps over the reversed arrows $s_x Q$ in both cases. The main result is that these functors are almost inverses to one another:

Theorem 2. *Let $x \in Q_0$ be a sink and V and indecomposable representation of Q . Then $C_x^+(V) = 0$ if and only if $V \simeq S_x$. If this is not the case, i.e. $V \not\simeq S_x$, then*

- (1) $C_x^+(V)$ is indecomposable
- (2) $C_x^- C_x^+(V) \cong V$
- (3) $\underline{\dim} C_x^+(V) = s_x \underline{\dim} V$

Similar results hold when x is a source.

The proofs are straightforward to follow out of Harm’s notes [Der01]. The book [ASS06] is also a good reference for the algebra-oriented.

Now we want to describe the representations of a fixed quiver Q , but these reflections functors change the quiver that we are working on. One solution is to go wild and learn tilting theory [ASS06] and prove that

the representation theory of Q and $s_x Q$ is “more or less” the same. We won’t go there. What we can do is the following: number the vertices of Q with $1, \dots, n$ so that all arrows go from bigger numbers to smaller ones, i.e. $ta > ha$ for every arrow $a \in Q_1$. Then the sequence of quiver reflections

$$c = c^+ := s_n s_{n-1} \cdots s_1$$

is “admissible”, meaning that after apply s_1, s_2 , and so forth up to s_i , the vertex $i + 1$ is always a sink so that we can now apply s_{i+1} . Since every arrow gets flipped exactly twice, $cQ = Q$. The analogously defined element $c \in W$ is called the **Coxeter element** of W (again we use the same notation for the reflection operation on the quiver as for the element of the reflection group W). This allows us to define an endofunctor on $\text{Rep}(Q)$ now,

$$C^+ := C_n^+ C_{n-1}^+ \cdots C_1^+ : \text{Rep}(Q) \rightarrow \text{Rep}(Q)$$

called the **Coxeter functor** on Q . (It is independent of choice of numbering.... not hard to show...)

A quick lemma (straight out of [Der01] that I won’t retype the proof of:

Lemma 3. *Let Q be a Dynkin quiver. Then for any non-zero dimension vector α , we have $c\alpha \neq \alpha$, and there exists a k such that $c^k \alpha$ is not nonnegative (i.e. some coordinate is negative).*

We now have developed enough tools to prove the other half of Gabriel’s theorem. So suppose Q is Dynkin, and let $V \in \text{Rep}(Q)$ be indecomposable of dimension α . Then $c^k \alpha$ is not nonnegative for some k , hence $(C^+)^k V = 0$ by the theorem. Thus some sequence of C_x^+ ’s sends V to 0, hence it must be sent to some simple S_y at some point (by theorem again). So the vector $\alpha \in E$ can be obtained by a sequence of reflections starting from ε_y (the inverse to whatever sequence took α to ε_y), hence α is a root.

But V must be the unique indecomposable of dimension vector α : for if W was another, the same sequence of reflection functors sends W to S_y , since the computation on the dimension vector is the same, and S_y is the unique representation of dimension ε_y . Applying the the corresponding C_\bullet^- operators, in the opposite order, takes S_y to both V and W , hence $V \simeq W$.

So the indecomposable representations are in bijection with the positive roots of the root system associated to the underlying graph of Q , hence there are finitely many of them since Q is Dynkin.

I might make some remarks about [DWZ07] but not feeling like typing anything about it.

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