

5.2 Rank of Matrix

- Row Space and Column Space

Let A be an $m \times n$ matrix.

- the **row space** of $A =$ the span of rows of $A \subset \mathbf{F}^n$
 $= \text{row}A$
- the **column space** of $A =$ the span of columns of $A \subset \mathbf{F}^m$
 $= \text{col}A$

Thm. $A : m \times n, U : p \times m, V : n \times q$

1. $\text{col}(AV) \subset \text{col}A$. If V is invertible, $\text{col}(AV) = \text{col}A$.
2. $\text{row}(UA) \subset \text{row}A$. If U is invertible, $\text{row}(UA) = \text{row}A$.

Proof.

$$V = [V_1 \ \cdots \ V_q], \quad AV = [AV_1 \ \cdots \ AV_q]$$

$$A = [C_1 \ \cdots \ C_n], \quad V_j = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

$$\begin{aligned} AV_j &= [C_1 \ \cdots \ C_n] \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \\ &= v_1 C_1 + v_2 C_2 + \cdots + v_n C_n \in \text{col}A \end{aligned}$$

$$\text{col}(AV) = \text{span}\{AV_1, \cdots, AV_q\} \subset \text{col}A$$

If V is invertible,

$$\text{col}A = \text{col}(AVV^{-1}) \subset \text{col}(AV).$$

Hence, $\text{col}A = \text{col}(AV)$.

$$\text{col}[(UA)^T] = \text{col}[A^T U^T] \subset \text{col}(A^T) \Leftrightarrow \text{row}(UA) \subset \text{row}A.$$

If U is invertible, U^T is invertible.

$$\text{col}[(UA)^T] = \text{col}[A^T U^T] = \text{col}(A^T) \Leftrightarrow \text{row}(UA) = \text{row}A.$$

□

Fact. Assume that $A \rightsquigarrow R$ in r.r.e.f.

1. The *nonzero* rows of R form a basis of $\text{row}R$.
2. The columns of R containing leading 1's form a basis of $\text{col}R$.

Thm. [Rank Thm]

$$\dim(\text{row}A) = \dim(\text{col}A) = \text{rank}A.$$

Moreover, suppose A is carried to R in r.r.e.f.

1. The *nonzero rows* of R form a basis of $\text{row}A$.
2. The columns of A corresponding leading 1's of R form a basis of $\text{col}A$.

Proof. As usual, $E_k \cdots E_1 A = R$, $U = E_k \cdots E_1$ is invertible, and $UA = R$.

$$\text{row}R = \text{row}(UA) = \text{row}A.$$

By Fact 1, we obtain the first part of the theorem.

Write $A = [C_1 \ \cdots \ C_n]$.

$$R = UA = U [C_1 \ \cdots \ C_n] = [UC_1 \ \cdots \ UC_n]$$

By Fact 2, the columns $UC_{j_1}, \cdots, UC_{j_r}$ of R containing leading 1's form a basis of $\text{col}R$. It is easy to see that $\{C_{j_1}, \cdots, C_{j_r}\}$ is linearly independent.

Similarly,

$$\begin{aligned} UC_j &= a_1UC_{j_1} + a_2UC_{j_2} + \cdots + a_rUC_{j_r} \\ &= U(a_1C_{j_1} + a_2C_{j_2} + \cdots + a_rC_{j_r}). \end{aligned}$$

Thus, $C_j = a_1C_{j_1} + a_2C_{j_2} + \cdots + a_rC_{j_r}$, and

$$\{C_1, C_2, \cdots, C_n\} \subset \text{span}\{C_{j_1}, \cdots, C_{j_r}\}.$$

Therefore, we have

$$\dim \text{row}A = \dim \text{col}A = r.$$

□

Eg.

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 2 & 1 & 3 \\ 7 & 1 & 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

1. $\text{rank}A = 2,$

2. a basis of $\text{row}A = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\},$

3. a basis of $\text{col}A = \left\{ \begin{bmatrix} 3 \\ 2 \\ 7 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}.$

Cor.

1. $\text{rank}A = \text{rank}A^T$.
2. *If $A : m \times n$, then $\text{rank}A \leq \min(m, n)$.*
3. $\text{rank}A = \text{rank}(UA) = \text{rank}(AV)$ where U and V are invertible.
4. *A is invertible if and only if $\text{rank}A = n$.*

- A Basis from a Spanning Set

Eg. Find a basis for $W = \text{span} \left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \\ 3 \\ -4 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 8 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 7 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \\ 4 \\ -3 \end{bmatrix} \right\}$.

1.

$$A = \begin{bmatrix} 1 & -2 & 0 & 3 & -4 \\ 3 & 2 & 8 & 1 & 4 \\ 2 & 3 & 7 & 2 & 3 \\ -1 & 2 & 0 & 4 & -3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Note that $W = \text{row}A$ and a basis of W is given by

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}.$$

2.

$$B = \begin{bmatrix} 1 & 3 & 2 & -1 \\ -2 & 2 & 3 & 2 \\ 0 & 8 & 7 & 0 \\ 3 & 1 & 2 & 4 \\ -4 & 4 & 3 & -3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{11}{24} \\ 0 & 1 & 0 & -\frac{49}{24} \\ 0 & 0 & 1 & \frac{7}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Note that $W = \text{col}A$ and another basis of W is given by

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \\ 3 \\ -4 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 8 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 7 \\ 2 \\ 3 \end{bmatrix} \right\}.$$

Eg.

$$\begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & -1 & 0 & 0 & 2 \\ 2 & 1 & 6 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\implies \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} x_1, x_2, x_4 : \text{leading variables} \\ x_3 = r, x_5 = s : \text{parameters} \end{cases}$$

$$X = \begin{bmatrix} -2r - s \\ -2r + s \\ r \\ -2s \\ s \end{bmatrix} = r \begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} : \text{basic solutions}$$

Thm. Let $A : m \times n$ with *rank* r .

1. If X_1, X_2, \dots, X_{n-r} are *basic solutions* of $AX = O$, then $\{X_1, \dots, X_{n-r}\}$ is a *basis* of $\text{null}A (= \ker A)$, and we have

$$\dim(\text{null}A) = n - r.$$

2. $\text{im}A = \text{col}A$, and

$$\dim(\text{im}A) = r = \text{rank}A.$$

Hence,

$$n = \dim(\text{null}A) + \dim(\text{im}A).$$

Thm. Assume that $A : m \times n$. Then A has *rank* r if and only if \exists invertible U and V s.t.

$$UAV = \begin{bmatrix} I_r & O \\ O & O \end{bmatrix},$$

where $I_r : r \times r$ identity matrix.

Proof. Assume A has rank r . Transform A into r.r.e.f. $UA = R$, and then transform R^T into r.r.e.f.

$$V'R^T = V'A^T U^T = \begin{bmatrix} I_r & O \\ O & O \end{bmatrix}.$$

Note that R and R^T have rank r . Letting $V' = V^T$, we obtain

$$UAV = \begin{bmatrix} I_r & O \\ O & O \end{bmatrix}.$$

□

– Algorithm for obtaining U and V

$$[A \ I] \Rightarrow U [A \ I] = [R \ U]$$

$$[R^T \ I] \Rightarrow V^T [R^T \ I] = \begin{bmatrix} I_r & O \\ O & O \end{bmatrix} V^T$$

Eg.

$$A = \begin{bmatrix} 1 & -1 & 1 & 2 \\ 2 & 2 & 1 & -1 \\ -1 & 1 & 0 & 3 \end{bmatrix}$$

Find U and V s.t. $UAV = \begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$ with $r = \text{rank}A$.

$$[A \quad I] \Rightarrow [R \quad U]$$

$$\begin{bmatrix} 1 & -1 & 1 & 2 & 1 & 0 & 0 \\ 2 & 2 & 1 & -1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 3 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 0 & -3 & 1 & 1 & 0 \\ 0 & 0 & 1 & 5 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & -1 & 0 & -3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 1 & 0 \\ 2 & -1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$$

$$[R^T \quad I] = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ -3 & 5 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & -5 & 1 \end{bmatrix}$$

$$V^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 3 & 0 & -5 & 1 \end{bmatrix}, \quad V = \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Def. Assume that A is given. We define

$$\overline{A} = [\overline{a_{ij}}], \text{ the conjugate of } A,$$

and

$$A^* = \overline{A^T}, \text{ the adjoint of } A.$$

Eg.

$$A = \begin{bmatrix} 4 + i & -2 + 3i \\ 6 + 4i & 3 \end{bmatrix},$$

$$\overline{A} = \begin{bmatrix} 4 - i & -2 - 3i \\ 6 - 4i & 3 \end{bmatrix}, \quad A^* = \begin{bmatrix} 4 - i & 6 - 4i \\ -2 - 3i & 3 \end{bmatrix}.$$

Prop.

$$1. A^* = \overline{A^T} = (\overline{A})^T$$

$$2. (AB)^* = B^* A^*$$

Def. We define

$$A : \textit{symmetric} \Leftrightarrow A^T = A$$

$$A : \textit{Hermitian} \Leftrightarrow A^* = A$$

Thm. Let A be an $m \times n$ matrix. T.F.A.E.

1. $AX = O$ has only the trivial solution.
2. The columns of A are linearly independent.
3. $\text{rank}A = n$
4. A^*A is invertible.

Proof. $1 \Leftrightarrow 2 \Leftrightarrow 3$: easy.

$4 \Rightarrow 1$: If $AX = O$, then $A^*AX = O$ so $X = O$.

1 \Rightarrow 4 : It suffices to show $(A^*A)X = O$ has only the trivial solution. Assume that $(A^*A)X = O$. Write $AX = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ and $(AX)^* = [\overline{y_1} \ \cdots \ \overline{y_n}]$.

$$\begin{aligned} 0 &= X^*A^*AX = (AX)^*AX \\ &= [\overline{y_1} \ \cdots \ \overline{y_n}] \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = |y_1|^2 + |y_2|^2 + \cdots + |y_n|^2 \end{aligned}$$

Thus $y_1 = y_2 = \cdots = y_n = 0$, and $AX = O$. By assumption we have $X = O$. \square

Thm. Let A be an $m \times n$ matrix. T.F.A.E.

1. $AX = B$ has a solution for every B .
2. The columns of A span \mathbf{F}^n .
3. $\text{rank}A = m$
4. AA^* is invertible.