Representation theory of \( p \)-adic groups and canonical bases

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Abstract

In this paper, we interpret the Gindikin–Karpelevich formula and the Casselman–Shalika formula as sums over Kashiwara–Lusztig’s canonical bases, generalizing the results of Bump and Nakasuji (2010) [7] to arbitrary split reductive groups. We also rewrite formulas for spherical vectors and zonal spherical functions in terms of canonical bases. In a subsequent paper Kim and Lee (preprint) [14], we will generalize these formulas to \( p \)-adic affine Kac–Moody groups.

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0. Introduction

This paper was inspired by a paper by Bump and Nakasuji [7]. Their basic philosophy is that an integral over a maximal unipotent subgroup of \( p \)-adic group can sometimes be replaced by a sum over crystal bases defined by Kashiwara [13]. The same approach was made by McNamara in [19]. They demonstrated this for the Gindikin–Karpelevich formula and the Casselman–Shalika formula for \( GL_n \). More precisely, let \( F \) be a \( p \)-adic field and \( N_- \) be the maximal unipotent subgroup of \( GL_n \). Let \( \chi \) be an unramified character of \( T \), the maximal
torus, and \( f^0 \) be the standard spherical vector corresponding to \( \chi \). Let \( z \) be the element of \( L^T \subset GL_n(\mathbb{C}) \), the \( L \)-group of \( GL_n \), corresponding to \( \chi \) by the Satake isomorphism. Then the Gindikin–Karpelevich formula for the longest Weyl group element can be written as

\[
\int_{N_-(F)} f^0(n) \, dn = \prod_{\alpha \in \Phi^+} \frac{1 - q^{-1}z^\alpha}{1 - z^\alpha} = \sum_{b \in \mathcal{B}} G_\Omega(b)q^{(\text{wt}(b), \rho)}z^{-\text{wt}(b)},
\]

(0.1)

where \( \mathcal{B} = \mathcal{B}(\infty) \) is the crystal basis for \( U^- \) (the negative part of the quantized enveloping algebra).

Let \( \lambda \) be a dominant integral weight and \( \chi_\lambda \) be the irreducible character of \( GL_n(\mathbb{C}) \) with the highest weight \( \lambda \). Then the Casselman–Shalika formula can be written as

\[
\int_{N_-(F)} f^0(n)\psi_\lambda(n) \, dn = \prod_{\alpha \in \Phi^+} \left(1 - q^{-1}z^\alpha\right)\chi_\lambda(z) = \sum_{b \in \mathcal{B}_\lambda} G_\Omega(b)q^{-\langle w_l(\text{wt}(b)), \rho \rangle}z^{\text{wt}(b)},
\]

(0.2)

where \( w_l \) is the longest Weyl group element.

The definition of the coefficients \( G_\Omega(b) \) in (0.1) and (0.2) is based on the “boxing rule” and “circling rule” in [4,5,7]. (See also [2,3,6].) We show that if we use canonical bases due to Lusztig [15] and tensor products of crystals, we obtain simple formulas for the coefficients in a uniform way. We can also generalize the above formulas. Namely, we prove, for any \( w \in W \) and for any split reductive group,

\[
\int_{N_w} f^0(w^{-1}n) \, dn = \prod_{\alpha \in \Phi(w)} \frac{1 - q^{-1}z^\alpha}{1 - z^\alpha} = \sum_{b \in G(w^{-1})} (1 - q^{-1})^{d(\phi(b))}z^{-\text{wt}(b)},
\]

(0.3)

\[
\prod_{\alpha \in \Phi(w)} (1 - q^{-1}z^\alpha)^{-1} = \sum_{b \in G(w^{-1})} q^{-\Sigma(\phi(b))}z^{-\text{wt}(b)},
\]

\[
\chi_\lambda(z) \prod_{\alpha \in \Phi^+} (1 - q^{-1}z^{-\alpha}) = z^{-\rho} \sum_{b' \in \mathcal{B}_\lambda \otimes \mathcal{B}_\rho} G_\rho(b; q)z^{\text{wt}(b' \otimes b)}.
\]

(0.4)

(See Propositions 1.4 and 2.1 for the notations.) Notice that in the Casselman–Shalika formula (0.4), we used crystal bases because they behave well with respect to the tensor product. Notice also that the left-hand side of (0.4) can be written as

\[
(-t)^M z^{-2\rho} \chi_\lambda(z) \prod_{\alpha \in \Phi^+} (1 - q^{-1}z^\alpha),
\]

where \( M = |\Delta_+| \) and \( t = q^{-1} \). Hence we obtain the product expansion of (0.2).

We first prove (0.3) by induction, and deduce (0.4) from (0.3) and the Weyl character formula. In the course of proof, we see that the Casselman–Shalika formula (0.4) can be considered as a \( q \)-deformation of the Weyl character formula.
K. Joshi and R. Raghunathan [12] constructed interesting infinite product identities for $L$-functions. As an application of our formulas, we write their identities in terms of canonical bases. In the subsequent paper [14], we will generalize all Gindikin–Karpelevich formula, we write the action of intertwining operators on some Iwahori-spherical functions due to Macdonald in terms of canonical bases. As a generalization of the bases (see Remark 2.21).

In Section 3, we interpret some formulas in [9] such as for spherical vectors and zonal spherical functions due to Macdonald in terms of canonical bases. As a generalization of the $\pi$ and $i$ will write $G$ and define $\Phi$ and define $Q$ elements $E_i$. Let $T_{\iota,1} = (i_1, i_2, \ldots, i_k)$ for $w \in W$ with the $k$-tuple $i=(i_1, i_2, \ldots, i_k)$. We will denote by $R(w)$ the set of all reduced expressions $i$ for $w$.

Let $U$ be the quantized enveloping algebra of $\mathfrak{g}$. Then $U$ is a $\mathbb{Q}(v)$-algebra generated by the elements $E_i, F_i, K_i^{\pm 1}, i \in \{1, 2, \ldots, r\}$. Let $U^+$ be the subalgebra generated by the $E_i$’s and let $U^-$ be the subalgebra generated by the $F_i$’s.

Let $T_{i_1,1}$ be the automorphism of $U$ as in Section 3.1.3 of [17]. For $c = (c_1, c_2, \ldots, c_k) \in \mathbb{N}^k$ and $i \in R(w)$, we let

$$F_i^c := F_{i_1}^{(c_1)} T_{i_1,1}'' \cdots T_{i_{k-1},1}'' T_{i_k,1}'' (F_{i_k}^{(c_k)}),$$

and define $G_i = \{ F_i^c : c \in \mathbb{N}^k \}$, and denote by $U^-_w$ the $\mathbb{Q}(v)$-span of $G_i$. Let $\sim : U \to U$ be the $\mathbb{Q}$-algebra automorphism of $U$ taking $E_i$ to $E_i$, $F_i$ to $F_i$, $K_i$ to $K_i^{-1}$, and $u \mapsto u^{-1}$.

**Theorem 1.1.** (See [15,18].) Suppose that $i \in R(w)$. The $\mathbb{Z}[v]$-span $L_w$ of $G_i$ is independent of $i$. Let $\pi : L_w \to L_w/\nu L_w$ be the natural projection. The image $\pi(G_i)$ is also independent of $i$; we denote it by $G_w$. The restriction of $\pi$ to $L_w \cap L_w^-$ is an isomorphism of $\mathbb{Z}$-modules $\pi_1 : L_w \cap L_w^- \to L_w/\nu L_w$, and $G(w) = \pi_1^{-1}(G_w)$ is a $\mathbb{Q}(v)$-basis of $U^-_w$.

When $w = w_i$, we obtain a $\mathbb{Q}(v)$-basis $G_{w_i}$ of $U^-$, which is called the canonical basis. We will write $B = G_{w_i}$. We define a map $\phi_i : G(w) \to \mathbb{N}^k$ for $i \in R(w)$ by setting $\phi_i(b) = c$, where $c \in \mathbb{N}^k$ is given by

$$b \equiv F_i^c \mod vL_w.$$ 

Then $\phi_i$ is a bijection.

For $w \in W$, we set

$$\Phi(w) = \{ \alpha \in \Phi_+ \mid w\alpha < 0 \}.$$

If $\ell(ws_i) > \ell(w)$, we have

$$\Phi(ws_i^{-1}) = \Phi(w^{-1}) \cup \{ w\alpha_i \}, \quad (1.2)$$

\[ \text{Author's personal copy} \]
and if \( \ell(s_i w) > \ell(w) \) then
\[
\Phi(w^{-1}s_i) = s_i \left( \Phi(w^{-1}) \right) \cup \{ \alpha_i \}. \tag{1.3}
\]

For \( \mathbf{c} = (c_1, c_2, \ldots, c_k) \in \mathbb{N}^k \), we denote by \( d(\mathbf{c}) \) the number of nonzero \( c_i \)'s.

**Proposition 1.4.** For any \( i \in R(w) \), \( w \in W \), we have
\[
\prod_{\alpha \in \Phi(siw^{-1})} \frac{1 - q^{-1}z^\alpha}{1 - z^\alpha} = \sum_{b \in G(w)} (1 - q^{-1})^{d(\phi_i(b))} z^{-wt(b)}. \tag{1.5}
\]

**Proof.** We will use induction on the length \( \ell(w) \) of \( w \). If \( w = s_i \) for some \( i \), then the identity (1.5) is easily verified. Assume that the identity (1.5) is true for \( w = s_{i_1} \cdots s_{i_k} \in W \), and that \( \ell(ws_i) = \ell(w) + 1 \). We will write \( i = (i_1, \ldots, i_k) \) and \( i' = (i_1, \ldots, i_k, i) \). Using (1.2) and an induction argument, we obtain
\[
\prod_{\alpha \in \Phi(ws_i^{-1})} \frac{1 - q^{-1}z^\alpha}{1 - z^\alpha} = \left( \prod_{\alpha \in \Phi(w^{-1})} \frac{1 - q^{-1}z^\alpha}{1 - z^\alpha} \right) \frac{1 - q^{-1}z^{ws_i \alpha_i}}{1 - z^{ws_i \alpha_i}}
\]
\[
= \left( \sum_{b \in G(w)} (1 - q^{-1})^{d(\phi_i(b))} z^{-wt(b)} \right) \left( 1 + \sum_{j \geq 1} (1 - q^{-1})z^{j ws_i \alpha_i} \right)
\]
\[
= \sum_{b \in G(w)} (1 - q^{-1})^{d(\phi_i(b))} z^{-wt(b)} + \sum_{j \geq 1} \sum_{b \in G(w)} (1 - q^{-1})^{d(\phi_i(b)) + 1} z^{-wt(b) + j ws_i \alpha_i}.
\]

On the other hand, since \( b' \in G(ws_i) \) satisfies
\[
b' \equiv b T_{t_1,-1} T_{t_2,-1} \cdots T_{t_k,-1} (F_i^{(j)}) \mod vL_{ws_i},
\]
for unique \( b \in G(w) \) and \( j \geq 0 \), we can write \( G(ws_i) \) as a disjoint union
\[
G(ws_i) = \bigcup_{j \geq 0} \{ b' \in G(ws_i) \mid \phi_{i'}(b') = (c_1, \ldots, c_k, j), \ c_i \in \mathbb{N} \}.
\]

Now it is clear that
\[
\sum_{b \in G(ws_i)} (1 - q^{-1})^{d(\phi_i(b))} z^{-wt(b)}
\]
\[
= \sum_{b \in G(w)} (1 - q^{-1})^{d(\phi_i(b))} z^{-wt(b)} + \sum_{j \geq 1} \sum_{b \in G(w)} (1 - q^{-1})^{d(\phi_i(b)) + 1} z^{-wt(b) + j ws_i \alpha_i}.
\]

This completes the proof. \( \square \)
Let $\widetilde{E}_i$ and $\widetilde{F}_i$ be the Kashiwara operators on $U^-$ as defined in [13]. Let $\mathcal{A} \subset \mathbb{Q}(v)$ be the subring of elements regular at $v = 0$, and let $\mathcal{L}'$ be the $\mathcal{A}$-lattice spanned by the set $S$ given by

$$S = \{ \widetilde{F}_{j_1} \widetilde{F}_{j_2} \cdots \widetilde{F}_{j_m} : 1 \in U^- \mid m \geq 0, \ j_k = 1, 2, \ldots, r \}.$$

**Theorem 1.6.** (See [13].)

(1) Let $\pi' : \mathcal{L}' \rightarrow \mathcal{L}' / v \mathcal{L}'$ be the natural projection, and let $B' = \pi'(S)$. Then $B'$ is a $\mathbb{Q}$-basis of $\mathcal{L}' / v \mathcal{L}'$, called the crystal basis.

(2) The operators $\widetilde{E}_i$ and $\widetilde{F}_i$ act on $\mathcal{L}' / v \mathcal{L}'$ for each $i = 1, 2, \ldots, r$. They satisfy

$$\widetilde{E}_i(B') \subseteq B' \cup \{ 0 \} \quad \text{and} \quad \widetilde{F}_i(B') \subseteq B'.$$

For $b, b' \in B'$ we have $\widetilde{F}_i b = b'$ if and only if $\widetilde{E}_i b' = b$.

(3) For each $b \in B'$, there is a unique element $\bar{b} \in \mathcal{L}' \cap \mathcal{L}'$ such that $\pi'(\bar{b}) = b$. The set of elements $\{ \bar{b} : b \in B' \}$ forms a basis of $U^-$, called the global basis of $U^-$. 

It was shown by Lusztig [16] that Kashiwara’s global basis coincides with his canonical basis. There is a parametrization of $\mathcal{B}$ arising from Kashiwara’s construction, and the parametrization again depends on $i \in R(w_l)$. Let $i = (i_1, i_2, \ldots, i_k) \in R(w_l)$ and $b \in \mathcal{B}$. Let $a_1$ be maximal such that $\widetilde{E}_{i_1} a_1 \neq 0$ mod $v \mathcal{L}'$; let $a_2$ be maximal such that $\widetilde{E}_{i_2} \widetilde{E}_{i_1} a_2 \neq 0$ mod $v \mathcal{L}'$, and so on. That is, $a_j$ is maximal such that $\widetilde{E}_{i_j} \widetilde{E}_{i_{j-1}} \cdots \widetilde{E}_{i_1} a_j \neq 0$ mod $v \mathcal{L}'$. We define a map $\psi_i : \mathcal{B} \rightarrow \mathbb{N}^k$ by $\psi_i(b) = a$, where $a = (a_1, a_2, \ldots, a_k) \in \mathbb{N}^k$ is determined as above. We obtain from Theorem 1.6(2) that $b \equiv \widetilde{F}_{i_1} \widetilde{F}_{i_2} \cdots \widetilde{F}_{i_k} - 1 \text{mod} \, v \mathcal{L}'$. The map $\psi_i$ is injective, and we will write $C_i = \psi_i(\mathcal{B})$. It is known that $C_i$ is a cone. For $i, j \in R(w_l)$, we define the Berenstein–Zelevinsky function $\sigma^i_j : C_i \rightarrow \mathbb{N}^k$ by

$$\sigma^i_j = \phi_j \psi_i^{-1}.$$

Descriptions of the cone $C_i$ and the function $\sigma^i_j$ can be obtained from Section 3 of [1].

For $a = (a_1, a_2, \ldots, a_k) \in \mathbb{N}^k$ and $i = (i_1, \ldots, i_k) \in R(w_l)$, we define

$$z^a = z^{a_1 \alpha_{i_1} + a_2 \alpha_{i_2} + \cdots + a_k \alpha_{i_k}}.$$

**Corollary 1.7.** For any $i, j \in R(w_l)$, we have

$$\prod_{\alpha \in \Phi_i} \frac{1 - q^{-1} z^a}{1 - z^a} = \sum_{a \in C_i} (1 - q^{-1})^d(\sigma^i_j(a)) z^a.$$

**Proof.** Considering the case $w = w_l$ in Proposition 1.4, we obtain the identity of the corollary from the definitions. \(\square\)

**Example 1.8.** Let $\mathfrak{g} = \mathfrak{sl}_4$. We choose $i = (1, 2, 1, 3, 2, 1) \in R(w_l)$. Then the cone $C_i$ is determined by
\[ a = (a_1, a_2, a_3, a_4, a_5, a_6) \in C_i \iff \begin{cases} a_1 \geq 0, & a_2 \geq a_3 \geq 0, \\ a_4 \geq a_5 \geq a_6 \geq 0, \end{cases} \]

and the Berenstein–Zelevinsky function \( \sigma_i^j : C_i \to \mathbb{N}^6 \) is given by

\[ \sigma_i^j(a) = (a_1, a_2, a_3, a_4, a_5, a_6) = (a_1, a_3 - a_2, a_6, a_4 - a_5, a_5 - a_6). \]

One can easily see that the “circling rule” for \( a_i \)'s in [5,7] is the same as having coordinates of \( \sigma_i^j(a) \) become zero. Then we obtain from Corollary 1.7 the result of Theorem 2 in Bump and Nakasuji’s paper [7].

2. Casselman–Shalika formula

For \( c = (c_1, c_2, \ldots, c_k) \in \mathbb{N}^k \), we set \( \Sigma(c) = c_1 + c_2 + \cdots + c_k \).

**Proposition 2.1.** For any \( i \in R(w) \) for \( w \in W \), we have

\[ \prod_{\alpha \in \Phi(w^{-1})} \left( 1 - q^{-1} z^{\alpha} \right)^{-1} = \sum_{b \in G(w)} q^{-\Sigma(\phi_i(b))} z^{-\mathrm{wt}(b)}. \]

**Proof.** One can prove the identity using an induction argument as in the proof of Proposition 1.4. We omit the details. \( \square \)

When \( w \) is a particular Weyl group element (see \( w_0 \) below), the left-hand side of the identity in Proposition 2.1 can be considered as a local \( L \)-function, and we have written the local \( L \)-function as a sum over a canonical basis. We make it more precise. We refer the reader to [10, Part II, Chapters 4 and 6], for the notations and relevant material. In particular, see Example 4.2 on p. 120 and Lemma 6.1 on p. 137.

Let \( \alpha \) be a simple root and set \( \theta = \Delta - \{\alpha\} \). Then \( \theta \) determines a maximal parabolic subgroup \( P = MN \), where \( M \) is the Levi subgroup. Let \( w_0 \) be the unique Weyl group element such that \( w_0(\alpha) < 0 \) and \( w_0(\theta) \subset \Delta \). Let \( \pi_p \) be a spherical representation of \( M(\mathbb{Q}_p) \) such that \( \pi_p \hookrightarrow I(\chi_p) \), where \( \chi_p \) is a quasi-character of the maximal split torus. Then for \( s \in \mathbb{C} \), we have \( I(s, \pi_p) \hookrightarrow I(\chi_p \otimes \exp(s\lambda, H_B(\cdot))) \), where \( \lambda \) is the fundamental weight corresponding to \( \alpha \). By Theorem 6.7 of [10], the identity (1.5) becomes

\[ \prod_{i=1}^m \frac{L(is, \pi_p, r_i)}{L(1 + is, \pi_p, r_i)} = \sum_{b \in G(w_0^{-1})} \left( 1 - p^{-1} \right)^{d(\phi_i(b))} z^{-\mathrm{wt}(b)}, \]

where \( z \) is the Satake parameter in \( L^A \subset L^G \) corresponding to the character \( \eta = \chi_p \otimes \exp(s\lambda, H_B(\cdot)) \) of \( A \) such that \( \eta \circ \beta^\vee = \beta^\vee(z) \) for any root \( \beta \). Here we are considering \( \beta^\vee \) on the right side as a root of \( L^G \).

Suppose that \( \pi_p \) is generic, i.e., it has a Whittaker model. Then by Theorem 8.11 of [10], the identity (2.2) becomes

\[ \prod_{i=1}^m L(1 + is, \pi_p, r_i) = \sum_{b \in G(w_0^{-1})} p^{-\Sigma(\phi_i(b))} z^{-\mathrm{wt}(b)}. \]
A special case is when $G = GL_{n+1}$ and $M = GL_n \times GL_1$. Consider the cuspidal representation $\pi \otimes \xi^{-1}$ of $M/\mathbb{Q}$, where $\pi = \bigotimes \pi_p$ is a cuspidal representation of $GL_n/\mathbb{Q}$ and $\xi = \bigotimes \xi_p$ is a Dirichlet character modulo $N$. Suppose $\pi_p \hookrightarrow I(\mu_1, \ldots, \mu_n)$. Then

$$I(s, \pi_p \otimes \xi^{-1}) \hookrightarrow I\left(\begin{array}{c} s \frac{\nu}{n+1} \\ \pi_1 \otimes \cdots \otimes \pi_n \end{array} \right| \nu \frac{\nu}{n+1} \otimes \xi^{-1}_p \left| - \frac{ns}{n+1} \right).$$

Hence $z = (\mu_1(p)p^{-s\frac{\nu}{n+1}}, \ldots, \mu_n(p)p^{-s\frac{\nu}{n+1}}, \xi^{-1}_p(p)p^{-\frac{ns}{n+1}})$. Therefore we have

$$L(s, \pi_p \times \xi_p) = \sum_{b \in G(0)} (1 - p^{-1})^d(\phi(b)) z^{-\nu(b)}$$

and

$$L(1 + s, \pi_p \times \xi_p) = \sum_{b \in G(0)} p^{-\Sigma(\phi(b))} z^{-\nu(b)}.$$

Let $P^+$ be the set of dominant integral weights and $Q^+ = \sum_{i=1}^r \langle Z_{\geq 0} \rangle \alpha_i$ be the $\mathbb{Z}_{\geq 0}$-span of $\Delta$. For $\lambda \in P^+$, let $L_\lambda$ be the irreducible highest weight module of $g$ with the highest weight $\lambda$.

**Definition 2.3.** Let $\lambda \in P^+$ and $i \in R(w_i)$. We define $H_\lambda(\cdot; q) : Q^+ \to \mathbb{Z}[q^{-1}]$ using the generating series

$$\sum_{\mu \in Q^+} H_\lambda(\mu; q) z^{\lambda - \mu} = \sum_{w \in W} (-1)^{\ell(w)} \sum_{b \in B} (1 - q^{-1})^d(\phi(b)) z^{w(\lambda + \nu) + \nu(b)},$$

and we write

$$\chi_q(L_\lambda) = \sum_{\mu \in Q^+} H_\lambda(\mu; q) z^{\lambda - \mu}.$$

In what follows, we will see that $H_\lambda(\cdot; q)$ does not depend on the choice of $i$.

We denote by $\chi(L_\lambda)$ the usual character of $L_\lambda$. (It was denoted by $\chi_{\lambda}$ in the Introduction.) By the Weyl character formula,

$$\sum_{w \in W} (-1)^{\ell(w)} z^{w(\lambda + \rho) - \rho} \prod_{\alpha \in \Phi_+} (1 - z^{-\alpha}) = \chi(L_\lambda).$$

In particular, if $\lambda = 0$, then

$$\sum_{w \in W} (-1)^{\ell(w)} z^{w \rho} = z^{\rho} \prod_{\alpha \in \Phi_+} (1 - z^{-\alpha}). \quad (2.4)$$
By Proposition 1.4,
\[ \sum_{b \in B} (1 - q^{-1})^{d(\phi_i(b))} z^{\text{wt}(b)} = \prod_{\alpha \in \Phi_+} \left( \frac{1 - q^{-1} z^{-\alpha}}{1 - z^{-\alpha}} \right). \]

Thus we obtain
\[
\chi_q(L_{\rho}) = \left( \sum_{w \in W} (-1)^{\ell(w)} z^{w_{\rho}} \right) \left( \sum_{b \in B} (1 - q^{-1})^{d(\phi_i(b))} z^{\text{wt}(b)} \right) \\
= z^{\rho} \prod_{\alpha \in \Phi_+} (1 - z^{-\alpha}) \prod_{\alpha \in \Phi_+} \left( \frac{1 - q^{-1} z^{-\alpha}}{1 - z^{-\alpha}} \right) \\
= z^{\rho} \prod_{\alpha \in \Phi_+} (1 - q^{-1} z^{-\alpha}).
\]

Therefore we have proved the following.

**Proposition 2.5.**
\[
\chi_q(L_{\rho}) = z^{\rho} \prod_{\alpha \in \Phi_+} (1 - q^{-1} z^{-\alpha}). \tag{2.6}
\]

When \( q = -1 \) in (2.6), we have the following identity.

**Lemma 2.7.**
\[
\chi_{-1}(L_{\rho}) = z^{\rho} \prod_{\alpha \in \Phi_+} (1 + z^{-\alpha}) = \chi(L_{\rho}).
\]

**Proof.** In (2.4), by replacing \( z^{\alpha_i} \) by \( z^{2\alpha_i} \) for each \( i = 1, \ldots, r \), we have
\[
\sum_{w \in W} (-1)^{\ell(w)} z^{2w_{\rho}} = z^{2\rho} \prod_{\alpha \in \Phi_+} (1 - z^{-2\alpha}).
\]

Since
\[
\chi(L_{\rho}) = \frac{\sum_{w \in W} (-1)^{\ell(w)} z^{2w_{\rho}}}{\sum_{w \in W} (-1)^{\ell(w)} z^{w_{\rho}}},
\]
we obtain the result. \( \Box \)

**Remark 2.8.** By Definition 2.3,
\[
\chi_{-1}(L_{\rho}) = \sum_{\mu \in Q^+} H_{\rho}(\mu; -1) z^{\rho - \mu} = z^{\rho} \prod_{\alpha \in \Phi_+} (1 + z^{-\alpha}).
\]
Therefore, if $H_\rho(\mu; -1) \neq 0$, $\rho - \mu$ must be a weight of $L_\rho$, and $H_\rho(\mu; -1)$ is the multiplicity of $\rho - \mu$ in $L_\rho$.

Now we have the following corollary which is a generalization of the second equality of the Casselman–Shalika formula (0.2).

**Corollary 2.9.**

\[
\chi_q(L_{\lambda + \rho}) = \chi(L_\lambda)\chi_q(L_\rho). \tag{2.10}
\]

**Proof.** By Definition 2.3 and Proposition 1.4,

\[
\chi_q(L_{\lambda + \rho}) = \left( \sum_{w \in W} (-1)^{\ell(w)} z^{w(\lambda + \rho)} \right) \prod_{\alpha \in \Phi^+} \left( \frac{1 - q^{-1} z^{-\alpha}}{1 - z^{-\alpha}} \right).
\]

By the Weyl character formula and Proposition 2.5, the right-hand side is $\chi(L_\lambda)\chi_q(L_\rho)$. $\square$

**Remark 2.11.** When $q = 1$, we see that $\chi_1(L_{\lambda + \rho})z^{-\rho}$ is the numerator of the Weyl character formula. Hence we can think of (2.10) as a $q$-deformation of the Weyl character formula. Since $\chi_\infty(L_\rho) = z^\rho$, by setting $q = \infty$, we have

\[
\chi_\infty(L_{\lambda + \rho}) = z^\rho \chi(L_\lambda).
\]

Hence by Definition 2.3,

\[
\sum_{\mu \in Q^+} H_{\lambda + \rho}(\mu; \infty)z^{\lambda - \mu} = \chi(L_\lambda).
\]

Therefore, $H_{\lambda + \rho}(\mu; \infty)$ is the multiplicity of the weight $\lambda - \mu$ in $L_\lambda$.

By putting $q = -1$ in (2.10), and by Lemma 2.7,

**Lemma 2.12.**

\[
\chi_{-1}(L_{\lambda + \rho}) = \sum_{\mu \in Q^+} H_{\lambda + \rho}(\mu; -1)z^{\lambda + \rho - \mu} = \chi(L_\lambda)\chi(L_\rho) = \chi(L_\lambda \otimes L_\rho).
\]

Hence, $H_{\lambda + \rho}(\mu; -1)$ is the multiplicity of the weight $\lambda + \rho - \mu$ in the tensor product $L_\lambda \otimes L_\rho$.

Before we further investigate the implication of the Casselman–Shalika formula (2.10), we need the following lemma.

**Lemma 2.13.** Assume that $\lambda_1, \lambda_2 \in P_+$. Then the set of weights of $L_{\lambda_1} \otimes L_{\lambda_2}$ is the same as that of $L_{\lambda_1 + \lambda_2}$. 

Proof. By Proposition 21.3, p. 114 in [11] μ is a weight of \( L_\lambda \) if and only if μ and all its Weyl conjugates are less than λ.

Now let \( \eta_1, \eta_2 \) be the weights of \( L_{\lambda_1}, L_{\lambda_2} \), respectively. Then all weights of \( L_{\lambda_1} \otimes L_{\lambda_2} \) are of the form \( \eta_1 + \eta_2 \) [11, p. 117, Exercise 7]. Hence it is enough to show that \( \eta_1 + \eta_2 \) is a weight of \( L_{\lambda_1 + \lambda_2} \). Then we are done, since it is clear that \( \eta_1 + \eta_2 \) and all its Weyl conjugates are less than \( \lambda_1 + \lambda_2 \). □

Now we use crystal bases, namely, bases at \( v = 0 \), since they behave nicely under tensor products. Let \( B_\lambda \) be the crystal basis associated to a dominant integral weight \( \lambda \in P_+ \). We choose \( G_\rho(\cdot; q) : B_\rho \rightarrow \mathbb{Z}[q^{-1}] \) by assigning any element of \( \mathbb{Z}[q^{-1}] \) to each \( b \in B_\rho \) so that

\[
H_\rho(\mu; q) = \sum_{b \in B_\rho} G_\rho(b; q) z^{wt(b)} = \sum_{b' \otimes b \in B_\lambda \otimes B_\rho} G_\rho(b; q) z^{wt(b'}). 
\]

(2.16)

Proof. The first equality is obvious from Proposition 2.5 and Corollary 2.9. For the second equality, we obtain

\[
\chi(L_\lambda) z^\rho \prod_{\alpha \in \Phi_+} (1 - q^{-1} z^{-\alpha}) = \chi(L_\lambda) \chi_q(L_\rho)
\]

\[
= \left( \sum_{b' \in B_\lambda} z^{wt(b')} \right) \left( \sum_{\mu \in Q^+} H_\rho(\mu; q) z^{\rho - \mu} \right)
\]

\[
= \left( \sum_{b' \in B_\lambda} z^{wt(b')} \right) \left( \sum_{b \in B_\rho} G_\rho(b; q) z^{wt(b)} \right)
\]

\[
= \sum_{b' \otimes b \in B_\lambda \otimes B_\rho} G_\rho(b; q) z^{wt(b')} \otimes b. 
\]

(2.16)

The following proposition provides useful information on \( H_{\lambda + \rho}(\mu; q) \in \mathbb{Z}[q^{-1}] \).

Proposition 2.17. Assume that \( \lambda \in P_+ \). Then we have \( H_{\lambda + \rho}(\mu; q) \) is a nonzero polynomial if and only if \( \lambda + \rho - \mu \) is a weight of \( L_{\lambda + \rho} \).
Proof. We obtain from (2.16) that if \( H_{\lambda+\rho}(\mu; q) \neq 0 \) then \( \lambda + \rho - \mu \) is a weight of \( L_\lambda \otimes L_\rho \). Then \( \lambda + \rho - \mu \) is a weight of \( L_{\lambda+\rho} \) by Lemma 2.13. Conversely, assume that \( \lambda + \rho - \mu \) is a weight of \( L_{\lambda+\rho} \), so a weight of \( L_\lambda \otimes L_\rho \). By Lemma 2.12,

\[
\sum_{\mu' \in Q^+} H_{\lambda+\rho}(\mu'; -1) z^{\lambda+\rho-\mu'} = \chi(L_\lambda \otimes L_\rho).
\]

Since \( \lambda + \rho - \mu \) is a weight of \( L_\lambda \otimes L_\rho \), the coefficient \( H_{\lambda+\rho}(\mu; -1) \neq 0 \). Then \( H_{\lambda+\rho}(\mu; q) \) is a nonzero polynomial.

Remark 2.18. We have proved that \( H_{\lambda+\rho}(\mu; q) = a_0 + a_1 q^{-1} + \cdots + a_k q^{-k} \), \( a_i \in \mathbb{Z} \), and \( H_{\lambda+\rho}(\mu; \infty) = a_0 \) is the multiplicity of the weight \( \lambda - \mu \) in \( L_\lambda \), and \( H_{\lambda+\rho}(\mu; -1) \) is the multiplicity of the weight \( \lambda + \rho - \mu \) in the tensor product \( L_\lambda \otimes L_\rho \). It would be interesting to study how \( H_{\lambda+\rho}(\mu; q) \) is related to the Kazhdan–Lusztig polynomials. We will pursue this in the subsequent paper [14].

Example 2.19. We consider the case \( g = sl_3 \) and fix \( i = (1, 2, 1) \in R(w_l) \). Using the circling and boxing rules in [5,7], we define \( G_{\rho}(b; q) \) for each \( b \in \mathfrak{B}_\rho \). Comparing Corollary 2.9 and Theorem 1 in Bump and Nakasuji’s paper [7], we see that the condition in (2.14) is satisfied.
We let $\lambda = \Lambda_1$ and consider $\mathcal{B}_\lambda \otimes \mathcal{B}_\rho$. We present a crystal graph of $\mathcal{B}_\lambda \otimes \mathcal{B}_\rho$ in the following figure. The tensor product should be read in the far-eastern reading in the figure. We put $G_\rho(b; q)$ for each $b' \otimes b \in \mathcal{B}_\lambda \otimes \mathcal{B}_\rho$. We can calculate $H_{\lambda + \rho}(\mu)$ by taking the sum of $G_\rho(b; q)$ over the crystals $b' \otimes b$ with $\text{wt}(b' \otimes b) = \lambda + \rho - \mu$, i.e.

$$H_{\lambda + \rho}(\mu; q) = \sum_{b' \otimes b \in \mathcal{B}_\lambda \otimes \mathcal{B}_\rho \atop \text{wt}(b' \otimes b) = \lambda + \rho - \mu} G_\rho(b; q).$$

If we define $G_{\lambda + \rho}(b; q)$ for $b \in \mathcal{B}_{\lambda + \rho}$ using the circling and boxing rules as in [5,7], it follows from Corollary 2.9 and Theorem 1 in [7] that

$$H_{\lambda + \rho}(\mu; q) = \sum_{b \in \mathcal{B}_{\lambda + \rho} \atop \text{wt}(b) = \lambda + \rho - \mu} G_{\lambda + \rho}(b; q).$$

**Remark 2.20.** Since we can define $G_\rho$ in an arbitrary way under the condition (2.14) and still obtain $H_{\lambda + \rho}$, the circling and boxing rules in [5,7] seem to be very special.

**Remark 2.21.** K. Joshi and R. Raghunathan [12] have interesting infinite product identities for the $L$-functions. We can interpret their identities in terms of canonical bases. Let $\pi = \bigotimes \pi_p$ be a cuspidal representation of $GL_n/\mathbb{Q}$. Given $N \in \mathbb{N}$, let $X_N$ be the set of all Dirichlet characters.
modulo $N$. Let $p$ be a fixed prime and $\xi_p$ be the local character at $p$ associated to $\xi \in X_N$ for some $N$. Since $\xi(p) = 0$ if $p|N$, Proposition 2.5 of [12] states

$$\frac{L(s, \pi_p)}{L(s + 1, \pi_p)} = \prod_{N=1}^{\infty} \prod_{\xi \in X_N} L(s + 1, \pi_p \times \xi_p).$$

Then by using Propositions 1.3 and 2.1, we can write the above identity in terms of canonical bases and obtain

$$\sum_{b \in G(w_0^{-1})} (1 - p^{-1})^d(\phi(b))z^{\text{wt}(b)} = \prod_{N=1}^{\infty} \prod_{\xi \in X_N} \left( \sum_{b \in G(w_0^{-1})} p^{-\Sigma(\phi(b))} z^{{\xi_p(p)}^{-1}} \right),$$

where $\pi_p \mapsto I(\mu_1, \ldots, \mu_n)$, and

$$z = (\mu_1(p)^{\frac{s}{n+1}}, \ldots, \mu_n(p)^{\frac{s}{n+1}}, p^{\frac{n}{n+1}}),$$

$$z_\xi = (\mu_1(p)^{\frac{s}{n+1}}, \ldots, \mu_n(p)^{\frac{s}{n+1}}, {\xi_p(p)}^{-1} p^{\frac{n}{n+1}}).$$

3. Spherical functions and generalization of the Gindikin–Karpelevich formula

Let us recall some notations and results from [9]. Let $G$ be a split reductive group. Let $O$ be the valuation ring of a $p$-adic field $F$ with its maximal ideal $P$ and $k = O/P$ be the residue field. We abuse notation and write $G = G(F)$. Let $K = G(O)$ and $I$ be the Iwahori subgroup of $K$ defined as the inverse image under the projection $G(O) \to G(k)$.

We define the $G$-projection $\mathcal{P}_\chi$ from $C^\infty_c$ onto $I(\chi)$ by

$$\mathcal{P}_\chi(f)(g) = \int_B \chi^{-1} \delta_B^\frac{1}{2}(b) f(bg) db.$$ 

For each $w \in W$, let $\phi_{w, \chi} = \mathcal{P}_\chi(1_{IwI})$, and let $\phi_{K, \chi} = \mathcal{P}_\chi(1_K)$. Here given a subset $S$ of $G$, the notation $1_S$ denotes the characteristic function of $S$. We will sometimes omit the reference to $\chi$. Then the functions $\phi_w$, $w \in W$, form a basis of $I(\chi)^I$, and the function $\phi_K$ is a basis of the 1-dimensional space $I(\chi)^K$.

The intertwining operator $T_w : I(\chi) \to I(w\chi)$ satisfies

$$T_w(\phi_{K, \chi}) = c_w(\chi) \phi_{K, w\chi},$$

where we set

$$c_w(\chi) = \prod_{\alpha \in \Phi(w)} c_\alpha(\chi) \quad \text{and} \quad c_\alpha(\chi) = \frac{1 - q^{-1}z^\alpha}{1 - z^\alpha}. $$

We have another basis $\{f_w\}$ of $I(\chi)^I$ such that
where we set \( N_w = (wNw^{-1} \cap N) \setminus N \). Then we have, for the longest Weyl group element \( w_l \),

\[
f_{w_l} = \phi_{w_l},
\]

and

\[
\phi_K = \sum_{w \in W} \phi_w = \sum_{w \in W} c_w(\chi) f_w.
\]

Using Proposition 1.4, we rewrite the above formula as follows.

**Proposition 3.1.**

\[
\phi_K = \sum_{w \in W} f_w \left( \sum_{b \in G(w^{-1})} (1 - q^{-1})^{d(\phi(b))} z^{-wt(b)} \right) = \sum_{b \in B} z^{-wt(b)} \left( \sum_{w \in W} (1 - q^{-1})^{d(\phi(b))} f_w \right).
\]

Now we study the effect of intertwining operators on Iwahori-fixed vectors. Casselman [9] proved that if \( \ell(s_i w) > \ell(w) \) for some \( i = 1, \ldots, r \), we have

\[
T_{s_i} \left( \phi_{w, \chi} \right) = (c_{s_i}(\chi) - 1) \phi_{w, s_i \chi} + q^{-1} \phi_{s_i w, s_i \chi}, \quad (3.2)
\]

\[
T_{s_i} \left( \phi_{s_i w, \chi} \right) = \phi_{w, s_i \chi} + (c_{s_i}(\chi) - q^{-1}) \phi_{w, s_i \chi}. \quad (3.3)
\]

One can find the following results in Reeder’s paper [20, p. 323].

**Lemma 3.4.**

1. If \( T_{x^{-1}}(\phi_w)(1) \neq 0 \), then \( w \leq x \).
2. \( T_{x^{-1}}(\phi_{w})(1) = 1 \).
3. If \( w \leq s_i w \), then \( T_{x^{-1}s_i}(\phi_w)(1) = c_{s_i}(\chi) - 1 \).

In the following proposition, we write the action of intertwining operators on some Iwahori-fixed vectors in terms of canonical bases. This can be regarded as a generalization of the Gindikin–Karpelevich formula. A relevant result and interesting conjectures can be found in [8].

**Proposition 3.5.** Assume that \( \ell(w'w) = \ell(w') + \ell(w) \) for \( w, w' \in W \). We write \( w' = s_i k s_{i-1} \cdots s_1 \) in a reduced expression and suppose that

\[
w \leq s_i j w, \quad \text{for each } j = 1, \ldots, k, \quad (3.6)
\]

\[
s_{i+1} w \not\leq s_i \cdots s_1 w, \quad \text{for each } j = 1, \ldots, k - 1. \quad (3.7)
\]
Then

\[ T_{w^{-1}w'}^{-1}(\phi_w)(1) = \prod_{\alpha \in \Phi(w^{-1})} \left( \frac{1 - q^{-1}z^\alpha}{1 - z^\alpha} - 1 \right) = (1 - q^{-1})^\ell(w') z^{\rho - w^{-1}\rho} \sum_{b \in G(w')} z^{-\text{wt}(b)}. \]

**Proof.** Using induction on the length of \( w' \), we first prove

\[ T_{w^{-1}w'}^{-1}(\phi_w)(1) = \prod_{\alpha \in \Phi(w^{-1})} \left( \frac{1 - q^{-1}z^\alpha}{1 - z^\alpha} - 1 \right). \tag{3.8} \]

If \( \ell(w') = 1 \), we obtain (3.8) from Lemma 3.4(3).

Now we consider the case \( \ell(w') = k \). Applying Casselman’s formulas (3.2), Lemma 3.4 and the assumptions (3.6), (3.7), we obtain

\[ T_{w^{-1}s_{i_1} \cdots s_{i_k}}(\phi_{w, \chi})(1) = T_{w^{-1}s_{i_1} \cdots s_{i_k-1}} T_{s_{i_k}}(\phi_{w, \chi})(1) \]

\[ = T_{w^{-1}s_{i_1} \cdots s_{i_k-1}} \left( (c_{s_{i_k}}(\chi) - 1) \phi_{w, s_{i_k}, \chi} + q^{-1} \phi_{s_{i_k}w, s_{i_k}, \chi} \right)(1) \]

\[ = (c_{s_{i_k}}(\chi) - 1) \prod_{\alpha \in \Phi(s_{i_1} \cdots s_{i_k-1})} \left( \frac{1 - q^{-1}z^{s_{i_k} \alpha}}{1 - z^{s_{i_k} \alpha}} - 1 \right) \]

\[ = \prod_{\alpha \in \Phi(w^{-1})} \left( \frac{1 - q^{-1}z^\alpha}{1 - z^\alpha} - 1 \right). \]

The last two equalities come from induction and (1.3), respectively.

We see that

\[ \prod_{\alpha \in \Phi(w^{-1})} \left( \frac{1 - q^{-1}z^\alpha}{1 - z^\alpha} - 1 \right) = (1 - q^{-1})^\ell(w') \prod_{\alpha \in \Phi(w^{-1})} \frac{z^\alpha}{1 - z^\alpha}. \tag{3.9} \]

Recall that we have

\[ \sum_{\alpha \in \Phi(w^{-1})} \alpha = \rho - w'^{-1}\rho. \]

Applying Proposition 2.1 to the right-hand side of (3.9) with \( q = 1 \), we obtain the second equality of the proposition. \( \Box \)

We give examples where the assumption (3.6) or (3.7) is not satisfied. We consider the root system of type \( C_2 \). Let \( \sigma = s_{\alpha_1}, \tau = s_{\alpha_2} \) be the simple reflections with respect to the short and long simple roots, respectively. Then the Weyl group is given by

\[ W = \{ 1, \sigma, \tau, \sigma \tau, \tau \sigma, \sigma \tau \sigma, \sigma \tau \sigma \tau = \tau \sigma \tau \sigma \}. \]
We first consider \( T_{\sigma \tau \sigma}(\phi_{\sigma, \chi})(1) \). In this case, \( \sigma \not\leq \sigma^2 \). So (3.6) is not satisfied, and we obtain
\[
T_{\sigma \tau \sigma}(\phi_{\sigma, \chi})(1) = T_{\sigma \tau}(\phi_{1, \sigma \chi} + (c_{\sigma}(\chi) - q^{-1})\phi_{\sigma, \sigma \chi})(1)
\]
\[
= (c_{\sigma \alpha_2}(\chi) - 1)(c_{\sigma \alpha_1}(\chi) - 1) + (c_{\sigma}(\chi) - q^{-1})(c_{\sigma \alpha_2}(\chi) - 1).
\]

Next we consider \( T_{\sigma \tau \sigma}(\phi_1)(1) \). Then \( \sigma \not\leq \tau \sigma \). So (3.7) is not satisfied, and we get
\[
T_{\sigma \tau \sigma}(\phi_1)(1) = T_{\sigma \tau}((c_{\sigma}(\chi) - 1)\phi_{1, \sigma \chi} + q^{-1}\phi_{\sigma, \sigma \chi})(1)
\]
\[
= \prod_{\alpha \in \Phi(\sigma \tau \sigma)} (c_{\alpha}(\chi) - 1) + q^{-1}(c_{\sigma \alpha_2}(\chi) - 1).
\]

In general, if the assumption (3.6) is satisfied and (3.7) is not, then
\[
T_{w^{-1}w^{-1}}(\phi_w)(1) = \prod_{\alpha \in \Phi(w^{-1})} (c_{\alpha}(\chi) - 1) + \text{some lower terms}.
\]

Let \( \Gamma_{\chi} \) be the zonal spherical function corresponding to \( \chi \). It satisfies
\[
\Gamma_{\chi}(1) = 1, \quad \Gamma_{\chi}(k_1 g k_2) = \Gamma_{\chi}(g) \quad \text{for all } k_1, k_2 \in K \text{ and } g \in G.
\]

We have the Cartan decomposition \( G = KT^{-1}K \), where \( T^{-1} \) corresponds to the set of dominant integral coweights. Then Macdonald’s identity shows that for \( t \in T^{-1} \), we have
\[
\Gamma_{\chi}(t) = Q^{-1} \delta_{B}^{1}(t) \sum_{w \in W} c_{w}( (w \chi)^{-1} (w \chi)(t),
\]
where \( Q = \sum_{w \in W} q^{-l(w)} \). Let \( z_{(w \chi)^{-1}} \) be the Satake parameter corresponding to \( (w \chi)^{-1} \). Then we see that \( z_{(w \chi)^{-1}} z_{w \chi}^{w t(b)} = z_{w \chi} w t(b) \). So by Proposition 1.4, we have
\[
c_{w}( (w \chi)^{-1} ) = \sum_{b \in B} (1 - q^{-1})^{d(\phi_i(b))} z_{w \chi} w t(b).
\]

Hence we obtain the following proposition.

**Proposition 3.10.**
\[
\Gamma_{\chi}(t) = Q^{-1} \delta_{B}^{1}(t) \sum_{w \in W} (w \chi)(t) \left( \sum_{b \in B} (1 - q^{-1})^{d(\phi_i(b))} z_{w \chi} w t(b) \right)
\]
\[
= Q^{-1} \delta_{B}^{1}(t) \sum_{b \in B} (1 - q^{-1})^{d(\phi_i(b))} \left( \sum_{w \in W} (w \chi)(t) z_{w \chi} w t(b) \right).
\]

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References


