WEYL GROUP MULTIPLE DIRICHLET SERIES FOR SYMMETRIZABLE KAC-MOODY ROOT SYSTEMS

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Abstract. In this paper, we construct Weyl group multiple Dirichlet series for root systems associated with symmetrizable Kac-Moody algebras, and establish their functional equations and meromorphic continuation.

Introduction

Weyl group multiple Dirichlet series were introduced in [2] by Brubaker, Bump, Chinta, Friedberg and Hoffstein, and have been studied in [1, 3, 4, 6]. Chinta and Gunnells also constructed these multiple Dirichlet series using different methods in [8, 9]. These multiple Dirichlet series are defined for a root system Φ of rank r and a global field F containing the $2n$-th roots of unity. These Dirichlet series are fundamental objects. They unify many examples in number theory that have been studied previously in a case-by-case basis, and are expected to be Whittaker coefficients of Eisenstein series on metaplectic groups. This expectation is now called *Eisenstein conjecture*, and some cases are already established. The development of these Dirichlet series inevitably involves representation theory, and Kashiwara's crystal bases turned out to be very useful tools [5].

In this paper, we generalize the construction of [9] given by Chinta and Gunnells and define Weyl group multiple Dirichelt series for all the root systems associated with symmetrizable Kac-Moody algebras. So Weyl groups are not finite, in general, and a root can have multiplicity bigger than one. Still, the results of this paper show that the generalization has all the standard properties: absolute convergence in a half plane, functional equations and meromorphic continuation. However, the meromorphic continuation is not to all of $\mathbb{C}^r$ but to the *Tits cone* as a consequence of standard facts in the Kac-Moody theory.

It is expected that one can also construct Weyl group multiple Dirichlet series for affine Kac-Moody root systems using crystal bases generalizing the results in [5]. Since there are various combinatorial objects that realize affine crystals, for example, *Young walls* ([10, 15]), one would be able to use these combinatorial tools to define the coefficients of the multiple Dirichlet series. At this point, one can ask: Are there any Eisenstein series corresponding to Kac-Moody root systems? We cannot clearly answer this question at present. However, we think that the Dirichlet series for affine Kac-Moody root system have connections to Garland's theory of Eisenstein series on loop groups [11, 12, 13]. We hope to return to these issues in the near future.

This paper consists of five sections. In the first section, we fix notations for root systems and a global field F. In Section 2, we define a Weyl group action on the field of Laurent series. In Section 3, the coefficients H in the Dirichlet series are defined. The next section is devoted to computations in the rank one case. In the last section, we collect the results of the previous sections and define Weyl group multiple Dirichlet series and prove their standard properties.

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1. Prelininaries

We refer the reader to [14] for a basic theory of Kac-Moody algebras. Let $A = (a_{ij})_{i,j=1}^r$ be an $r \times r$ symmetrizable generalized Cartan matrix of rank l, and $(\mathfrak{h}, \Delta, \Delta^\vee)$ be a realization of A, where $\Delta = \{\alpha_1, ..., \alpha_r\} \subset \mathfrak{h}^*$ and $\Delta^\vee = \{h_1, ..., h_r\} \subset \mathfrak{h}$ such that

$$\alpha_j(h_i) = a_{ij}, \qquad i, j = 1, ..., r.$$

Let $\mathfrak{g}(A)$ be the symmetrizable Kac-Moody algebra associated to $(\mathfrak{h}, \Delta, \Delta^\vee)$. Then we have $\dim \mathfrak{h} = 2r - l$. We denote by Φ the set of roots of $\mathfrak{g}(A)$ and have $\Phi = \Phi_+ \cup \Phi_-$ where Φ_+ (resp. Φ_-) is the set of positive (resp. negative) roots. Denote by $\Phi^{\rm re}$ (resp. $\Phi^{\rm im}$) the set of real (resp. imaginary) roots, and put $\Phi^{\rm re}_\pm = \Phi^{\rm re} \cap \Phi_\pm$ and $\Phi^{\rm im}_\pm = \Phi^{\rm im} \cap \Phi_\pm$. We fix a decomposition

$$A = \mathrm{diag}(\epsilon_1, \cdots, \epsilon_r) B, \tag{1.1}$$

where ϵ_i are positive rational numbers and $B = (b_{ij})$ is a symmetric half-integral matrix, i.e. $b_{ij} = b_{ji}$, $b_{ii} \in \mathbb{Z}$ and $b_{ij} \in \frac{1}{2}\mathbb{Z}$. We will write $b_i = b_{ii}$. As in Chapter 2 of [14], we define a standard symmetric bilinear form (,) on $\mathfrak{h}^*$ such that

$$(\alpha_i, \alpha_j) = b_{ij} \qquad \text{for } i, j = 1, ..., r.$$

We extend the sets Δ and $\Delta^\vee$, and choose bases

$$\Delta \cup \{\delta_k \mid k = 1, ..., r - l\} \quad \text{and} \quad \Delta^\vee \cup \{d_k \mid k = 1, ..., r - l\}$$

for $\mathfrak{h}^*$ and $\mathfrak{h}$, respectively, such that

$$\alpha_j(d_k) = 0 \text{ or } 1, \quad \delta_k(h_j) = 0 \text{ or } 1 \quad \text{and} \quad \delta_k(d_{k'}) = 0$$

for $j = 1, ..., r$ and $k, k' = 1, ..., r-l$. Let $P^\vee$ be the $\mathbb{Z}$-span of the basis $\Delta^\vee \cup \{d_k \mid k = 1, ..., r-l\}$, and let

$$\mathfrak{h}_\mathbb{R} = \mathbb{R} \otimes P^\vee \subset \mathfrak{h}.$$

We define

$$P = \{\lambda \in \mathfrak{h}^* | \lambda(P^\vee) \subseteq \mathbb{Z}\},$$

and

$$P_+ = \{\lambda \in \mathfrak{h}^* | \lambda(P^\vee) \subseteq \mathbb{Z}_{\geq 0}\}.$$

Define $\omega_i \in \mathfrak{h}^*$ $(i = 1, ..., r)$ by

$$\omega_i(h_j) = \delta_{ij}, \quad \omega_i(d_k) = 0, \qquad j = 1, ..., r, \qquad k = 1, ..., r - l$$

and put

$$\rho = \sum_{i=1}^{r} \omega_i.$$

Similarly, we define $\omega_i^\vee \in \mathfrak{h}$ $(i = 1, ..., r)$ by

$$\alpha_j(\omega_i^\vee) = \delta_{ij}, \quad \delta_k(\omega_i^\vee) = 0, \qquad j = 1, ..., r, \qquad k = 1, ..., r - l$$

and put

$$\rho^\vee = \sum_{i=1}^{r} \omega_i^\vee.$$

Define

$$Q = \bigoplus_{i=1}^{r} \mathbb{Z}\alpha_i \quad \text{and} \quad Q_+ = \bigoplus_{i=1}^{r} (\mathbb{Z}_{\geq 0})\alpha_i.$$

We also define

$$Q^\vee = \bigoplus_{i=1}^{r} \mathbb{Z} h_i. \tag{1.2}$$

We have the usual ordering on P (and on Q) given by

$$\lambda > \mu \;\Leftrightarrow \lambda - \mu \in Q_+.$$

For $\beta \in Q$, we write $\beta = k_1\alpha_1 + \cdots + k_r\alpha_r$ and define

$$d(\beta) = \beta(\rho^\vee) = k_1 + \cdots + k_r.$$

Let W be the Weyl group of $\mathfrak{g}(A)$ generated by the simple reflections $\sigma_i \in GL(\mathfrak{h}^*)$. We have the standard actions of W on $\mathfrak{h}$ and on $\mathfrak{h}^*$. For $w \in W$, we let

$$\Phi(w) = \Phi_+ \cap w^{-1}\Phi_- \subseteq \Phi_+^{\mathrm{re}}.$$

We denote by $\ell(w)$ the length of w, and define $\mathrm{sgn}(w) = (-1)^{\ell(w)}$. If $\ell(\sigma_i w) = \ell(w) + 1$ then

$$\Phi(\sigma_i w) = \Phi(w) \cup \{w^{-1}\alpha_i\}, \tag{1.3}$$

and if $\ell(w\sigma_i) = \ell(w) + 1$ then

$$\Phi(w\sigma_i) = \sigma(\Phi(w)) \cup \{\alpha_i\}. \tag{1.4}$$

Let $n > 1$ be an integer and let F be an algebraic number field that contains the group μ_{2n} of $2n$-th roots of unity. Let S be a finite set of places of F containing the archimedean ones, all those which are ramified over $\mathbb{Q}$ and enough others so that the ring $\mathfrak{o}_S$ of S-integers is a principal ideal domain. We embed $\mathfrak{o}_S$ into $F_S = \prod_{v\in S} F_v$ along the diagonal. We choose a nontrivial additive character ψ of F_S such that $\psi(x\mathfrak{o}_S) = 1$ if and only if $x \in \mathfrak{o}_S$. Let S_∞ be the set of archimedean places in S, and S_{fin} be the set of the nonarchimedean places so that $S = S_\infty \cup S_{\mathrm{fin}}$. We denote

$$F_\infty = \prod_{v\in S_\infty} F_v \qquad \text{and} \qquad F_{\mathrm{fin}} = \prod_{v \in S_{\mathrm{fin}}} F_v.$$

Then $F_S = F_\infty \times F_{\mathrm{fin}}$. Let $(x, y)_S = \prod_{v\in S}(x, y)_v$ be the S-Hilbert symbol, where we take the same convention on $(\ ,\ )_v$ as in [3], i.e. it is the inverse of the symbol used in [16]. If c, d are coprime elements of $\mathfrak{o}_S$, let $(\frac{c}{d})$ denote the n-th power residue symbol. Then we have the reciprocity law

$$\left(\frac{c}{d}\right) = (d,c)_S \left(\frac{d}{c}\right).$$

We fix an isomorphism $\epsilon : \mu_n \to \{x \in \mathbb{C}^\times \mid x^n = 1\}$ and will suppress this isomorphism from the notation. If t is any positive integer and $a, c \in \mathfrak{o}_S$, we define the Gauss sum

$$g(a,c;t) = \sum_{d \bmod c} \left(\frac{d}{c}\right)^t \psi\left(\frac{ad}{c}\right).$$

For $\mathbf{x}, \mathbf{y} \in (F_S^\times)^r$ and for each i, we define

$$(\mathbf{x},\mathbf{y})_{S,i}^B = (x_i, y_i)_S^{b_i} \prod_{j>i} (x_i, y_j)_S^{2b_{ij}},$$

where $\mathbf{x} = (x_1, \cdots, x_r)$ and $\mathbf{y} = (y_1, \cdots, y_r)$, and set

$$(\mathbf{x},\mathbf{y})_S^B = \prod_{i=1}^r (\mathbf{x},\mathbf{y})_{S,i}^B.$$

We also define

$$\xi_B(\mathbf{x},\mathbf{y}) = \prod_{i=1}^r \left(\frac{x_i}{y_i}\right)^{b_i} \left(\frac{y_i}{x_i}\right)^{b_i} \prod_{i<j} \left(\frac{x_i}{y_j}\right)^{2b_{ij}} \left(\frac{y_i}{x_j}\right)^{2b_{ij}},$$

$$\left[\frac{\mathbf{x}}{\mathbf{y}}\right]^B = \prod_{i=1}^r \left(\frac{x_i}{y_i}\right)^{b_i} \qquad \text{and} \qquad \left[\frac{\mathbf{x}}{\mathbf{y}}\right]^{-B} = \prod_{i=1}^r \left(\frac{x_i}{y_i}\right)^{-b_i}$$

when $\mathbf{x}, \mathbf{y} \in (F_S^\times)^r \cap (\mathfrak{o}_S)^r$.

Let $\Omega = \mathfrak{o}_S^\times F_S^{\times,n}$ where $F_S^{\times,n}$ is the subgroup of n-th powers in $F_S^\times$. Let $\mathcal{M}_B(\Omega)$ be the space of functions $\Psi : (F_S^\times)^r \to \mathbb{C}$ such that

$$\Psi(\mathbf{e}\mathbf{c}) = (\mathbf{e},\mathbf{c})_S^B \; \Psi(\mathbf{c})$$

when $\mathbf{e} \in \Omega^r$ and $\mathbf{c} \in (F_{\text{fin}}^\times)^r$. If $r = 1$ and $B = (t)$ we simply write $\mathcal{M}_B(\Omega) = \mathcal{M}_t(\Omega)$.

2. Weyl Group Actions on Laurent Series

Let $\mathcal{A} = \mathbb{C}[Q]$ be the group algebra of the lattice Q. An element $f \in \mathcal{A}$ can be written as $f = \sum_{\beta \in Q} c_\beta \, \mathbf{x}^\beta$ ($c_\beta \in \mathbb{C}$) with almost all c_β zero. We identify $\mathcal{A}$ with $\mathbb{C}[x_1^{\pm 1}, \cdots, x_r^{\pm 1}]$ via $\mathbf{x}^{\alpha_i} \mapsto x_i$. Let $\widetilde{\mathcal{A}}$ be the field of fractions of $\mathcal{A}$. We also let $\mathcal{B} = \mathbb{C}[[x_1, \cdots, x_r]]$ be the ring of power series. For $\lambda \in Q$ we set

$$D(\lambda) = \{\mu \in Q \mid \mu \le \lambda\}.$$

We define a ring $\mathcal{E}$. The elements of $\mathcal{E}$ are series of the form

$$\sum_{\beta \in Q} c_\beta \, \mathbf{x}^\beta$$

where $c_\beta \in \mathbb{C}$ and $c_\beta = 0$ for β outside the union of a finite number of sets of the form $D(\mu)$. The addition and multiplication are defined in an obvious way. We let $\widetilde{\mathcal{E}}$ be the field of fractions of $\mathcal{E}$. We regard $\widetilde{A}$ as a subfield of $\widetilde{\mathcal{E}}$. Then we have

$$\mathcal{A} \subset \widetilde{\mathcal{A}} \subset \widetilde{\mathcal{E}} \quad \text{and} \quad \mathcal{B} \subset \mathcal{E} \subset \widetilde{\mathcal{E}}.$$

Given $f = \sum_{\beta \in Q} c_\beta \, \mathbf{x}^\beta \in \mathcal{E}$, we define

$$\mathrm{supp}(f) = \{\beta \in Q \mid c_\beta \neq 0\},$$

and define

$$\mathrm{val}(f) = \min\{d(\beta) \in \mathbb{Z} \mid \beta \in \mathrm{supp}(f)\}.$$

Let q be a positive integer. We consider a collection of complex numbers $\gamma(i) \in \mathbb{C}$, indexed by the integers modulo n, and such that $\gamma(0) = -1$ and

$$\gamma(i)\gamma(-i) = 1/q \qquad \text{if } \; i \not\equiv 0 \mod n.$$

We also define

$$m(\alpha) = \frac{n}{\gcd(n, (\alpha, \alpha))} \qquad \text{for each } \alpha \in \Phi.$$

We write $\mathbf{x} = (x_1, ..., x_r)$, and define a change-of-variable formula by

$$(\sigma_i \mathbf{x})_j = (q x_i)^{-a_{ij}} x_j$$

for a simple reflection $\sigma_i \in W$, where $A = (a_{ij})$ is the generalized Cartan matrix. One can check that this definition extends to the whole group W. For $\beta = \sum k_i \alpha_i \in Q$ and $w\mathbf{x} = (y_1, ..., y_r)$, $w \in W$, we define

$$(w\mathbf{x})^{\beta} = y_1^{k_1} \cdots y_r^{k_r}.$$

Then we have

$$(w\mathbf{x})^{\beta} = q^{d(w^{-1}\beta - \beta)} \mathbf{x}^{w^{-1}\beta} \quad \text{for } w \in W. \tag{2.1}$$

In particular, we obtain

$$(\sigma_i \mathbf{x})^{\beta} = (qx_i)^{-\beta(h_i)} \mathbf{x}^{\beta}.$$

In the rest of this section, we fix $\lambda \in P$. We define a shifted action of W by

$$\sigma_i \cdot \beta = \sigma_i(\beta - \lambda - \rho) + \lambda + \rho, \qquad \beta \in Q, \quad i = 1, ..., r.$$

For any $\beta \in Q$, we set

$$\mu_{i,\lambda}(\beta) = \mu_i(\beta) = (\lambda + \rho - \beta)(h_i), \qquad i = 1, ..., r.$$

Then we have

$$\sigma_i \cdot \beta = \beta + \mu_i(\beta)\alpha_i = \sigma_i \beta + \mu_i(0)\alpha_i.$$

Let $Q' \subseteq Q$ be the sublattice of Q generated by the set $\{m(\alpha)\alpha \mid \alpha \in \Phi\}$. Since $m(w\alpha) = m(\alpha)$ for $w \in W$ and $\alpha \in \Phi$, the sublattice Q' is preserved under the action of W on Q. Let $\nu : Q \to Q/Q'$ be the projection. We have the decomposition

$$\widetilde{\mathcal{A}} = \bigoplus_{\lambda \in Q/Q'} \widetilde{A}_{\lambda}$$

where $\widetilde{\mathcal{A}}_{\lambda}$ consists of functions f/g ($f, g \in \mathcal{A}$) such that $\nu(\text{supp } g) = 0$ and $\nu(\text{supp } f) = \lambda$. Thus an element of $\widetilde{\mathcal{A}}$ can be written a finite sum of terms of the form $f(\mathbf{x})\mathbf{x}^{\beta}$ with $f(\mathbf{x}) \in \widetilde{\mathcal{A}}_0$ and $\beta \in Q$. We also define

$$\mathcal{B}_0 = \{f \in \mathcal{B} \mid \text{supp}(f) \subset Q'\} \quad \text{and} \quad \widetilde{\mathcal{B}}_0 = \{f \in \widetilde{\mathcal{B}} \mid \text{supp}(f) \subset Q'\}.$$

Now we define the action of W on $\widetilde{\mathcal{A}}$ for a generator $\sigma_i \in W$. Put $m = m(\alpha_i)$. For an integer k, we denote by $(k)_m$ the remainder upon division of k by m and by $[k]_m$ be the largest multiple of m that is smaller than or equal to k. We define, for any $\beta \in Q$,

$$\mathcal{P}_{\beta,i}(\mathbf{x}) = (q\mathbf{x}^{\alpha_i})^{[\mu_i(\beta)]_m} \frac{1-1/q}{1-(q\mathbf{x}^{\alpha_i})^m/q},$$

and

$$\mathcal{Q}_{\beta,i}(\mathbf{x}) = \gamma(b_i\mu_i(\beta))q^{\mu_i(\beta)} \frac{1-(q\mathbf{x}^{\alpha_i})^{-m}}{1-(q\mathbf{x}^{\alpha_i})^m/q}.$$

Note that $\mathcal{P}_{\beta,i}(\mathbf{x})$ and $\mathcal{Q}_{\beta,i}(\mathbf{x})$ belong to $\widetilde{\mathcal{A}}_0$.

Definition 2.2. For $\beta \in Q$ and each $i = 1, ..., r$, we define

$$\mathbf{x}^\beta|_\lambda \sigma_i = \mathcal{P}_{\beta,i}(\mathbf{x})\mathbf{x}^\beta + \mathcal{Q}_{\beta,i}(\mathbf{x})\mathbf{x}^{\sigma_i\cdot\beta}.$$

Assume that $g(\mathbf{x}) \in \widetilde{\mathcal{A}}_0$. Then we define

$$(g\mathbf{x}^\beta|_\lambda \sigma_i)(\mathbf{x}) = g(\sigma_i\mathbf{x})(\mathbf{x}^\beta|_\lambda \sigma_i)(\mathbf{x})$$

and extend this action linearly to all of $\widetilde{\mathcal{A}}$.

Remark 2.3. The definitions of $\mathcal{P}_{\beta,i}$ and $\mathcal{Q}_{\beta,i}$ are slightly different from those in [9], but the definition of the action of σ_i is the same as in [9].

Theorem 2.4. *The action of generators σ_i defined in Definition 2.3 extends to give an action of W on $\widetilde{\mathcal{A}}$.*

Proof. The group W is a Coxeter group. Thus W is generated by σ_i $(i = 1, ..., r)$ and the defining relations are

$$\sigma_i^2 = 1, \qquad (\sigma_i\sigma_j)^{m_{ij}} = 1 \qquad \text{for } i, j = 1, ..., r,$$

where

$$m_{ij} = \begin{cases} 2 & \text{if } a_{ij}a_{ji} = 0; \\ 3 & \text{if } a_{ij}a_{ji} = 1; \\ 4 & \text{if } a_{ij}a_{ji} = 2; \\ 6 & \text{if } a_{ij}a_{ji} = 3; \\ \infty & \text{if } a_{ij}a_{ji} \geq 4. \end{cases}$$

Here $x^\infty = 1$ for any x by notational convention. Therefore, we need only to consider the following four cases:

$$\begin{cases} a_{ij} = 0, & a_{ji} = 0, \quad b_i \text{ and } b_j \text{ are arbitrary;} \\ a_{ij} = -1, & a_{ji} = -1, \quad b_i = b_j; \\ a_{ij} = -1, & a_{ji} = -2, \quad b_i = 2b_j; \\ a_{ij} = -1, & a_{ji} = -3, \quad b_i = 3b_j. \end{cases}$$

$\square$

Remark 2.5. We obtain, from the definition,

$$(gf|_\lambda w)(\mathbf{x}) = g(w\mathbf{x})(f|_\lambda w)(\mathbf{x}) \quad \text{for } g \in \widetilde{\mathcal{A}}_0 \text{ and } w \in W.$$

We denote the multiplicity of $\alpha \in \Phi$ by $\mathrm{mult}(\alpha)$ and define

$$\Delta(\mathbf{x}) = \prod_{\alpha\in\Phi_+} (1 - q^{m(\alpha)d(\alpha)}\mathbf{x}^{m(\alpha)\alpha})^{\mathrm{mult}(\alpha)} \quad \text{and} \quad D(\mathbf{x}) = \prod_{\alpha\in\Phi_+} (1 - q^{m(\alpha)d(\alpha)-1}\mathbf{x}^{m(\alpha)\alpha})^{\mathrm{mult}(\alpha)}.$$

Note that $\Delta(\mathbf{x}), D(\mathbf{x}) \in \mathcal{B}$.

Lemma 2.6. (1) *For $w \in W$, we obtain $\Delta(w\mathbf{x}) \in \widetilde{\mathcal{B}}_0$ and*

$$j(w, \mathbf{x}) := \Delta(\mathbf{x})/\Delta(w\mathbf{x}) = \mathrm{sgn}(w)q^{d(\beta)}\mathbf{x}^\beta,$$

where

$$\beta = \sum_{\alpha\in\Phi(w)} m(\alpha)\alpha.$$

(2) *The function $j(w, \mathbf{x})$ satisfies the cocycle relation*

$$j(ww', \mathbf{x}) = j(w, w'\mathbf{x})j(w', \mathbf{x}).$$

Proof. (1) Since $m(w\alpha) = m(\alpha)$ and $\mathrm{mult}(w\alpha) = \mathrm{mult}(\alpha) = \mathrm{mult}(-\alpha)$, we have

$$\begin{aligned}
\Delta(w\mathbf{x}) &= \prod_{\alpha\in\Phi_+} (1 - q^{m(\alpha)d(\alpha)}(w\mathbf{x})^{m(\alpha)\alpha})^{\mathrm{mult}(\alpha)} \\
&= \prod_{\alpha\in\Phi_+} (1 - q^{m(\alpha)d(w^{-1}\alpha)}\mathbf{x}^{m(\alpha)(w^{-1}\alpha)})^{\mathrm{mult}(\alpha)} \qquad \text{(usng (2.1))} \\
&= \prod_{\substack{\alpha\in\Phi_+ \\ w^{-1}\alpha\in\Phi_-}} (1 - q^{m(\alpha)d(w^{-1}\alpha)}\mathbf{x}^{m(\alpha)(w^{-1}\alpha)})^{\mathrm{mult}(\alpha)} \\
&\qquad\qquad \times \prod_{\substack{\alpha\in\Phi_+ \\ w^{-1}\alpha\in\Phi_+}} (1 - q^{m(\alpha)d(w^{-1}\alpha)}\mathbf{x}^{m(\alpha)(w^{-1}\alpha)})^{\mathrm{mult}(\alpha)} \\
&= \prod_{\substack{\alpha\in\Phi_- \\ w\alpha\in\Phi_+}} (1 - q^{m(\alpha)d(\alpha)}\mathbf{x}^{m(\alpha)\alpha})^{\mathrm{mult}(\alpha)} \prod_{\substack{\alpha\in\Phi_+ \\ w\alpha\in\Phi_+}} (1 - q^{m(\alpha)d(\alpha)}\mathbf{x}^{m(\alpha)\alpha})^{\mathrm{mult}(\alpha)} \\
&= \prod_{\alpha\in\Phi(w)} (1 - q^{-m(\alpha)d(\alpha)}\mathbf{x}^{-m(\alpha)\alpha})^{\mathrm{mult}(\alpha)} \prod_{\substack{\alpha\in\Phi_+ \\ w\alpha\in\Phi_+}} (1 - q^{m(\alpha)d(\alpha)}\mathbf{x}^{m(\alpha)\alpha})^{\mathrm{mult}(\alpha)}.
\end{aligned}$$

Since $|\Phi(w)| = \ell(w) < \infty$, we see that $\Delta(w\mathbf{x}) \in \widetilde{\mathcal{B}}_0$. It follows from $\Phi(w) \subset \Phi^{\mathrm{re}}$ that $\mathrm{mult}(\alpha) = 1$ for each $\alpha \in \Phi(w)$. Now we have

$$\begin{aligned}
j(w, \mathbf{x}) &= \frac{\Delta(\mathbf{x})}{\Delta(w\mathbf{x})} = \prod_{\alpha\in\Phi(w)} \frac{1 - q^{m(\alpha)d(\alpha)}\mathbf{x}^{m(\alpha)\alpha}}{1 - q^{-m(\alpha)d(\alpha)}\mathbf{x}^{-m(\alpha)\alpha}} \\
&= \prod_{\alpha\in\Phi(w)} - q^{m(\alpha)d(\alpha)}\mathbf{x}^{m(\alpha)\alpha} \\
&= \mathrm{sgn}(w) q^{d(\beta)}\mathbf{x}^{\beta}.
\end{aligned}$$

(2) It is straightforward to verify the identity. □

Lemma 2.7. *Let $\lambda \in P_+$ and $\beta \in Q$.*

(1) *The function*

$$(w\mathbf{x})^{-\beta} j(w, \mathbf{x})(\mathbf{x}^{\beta}|_{\lambda} w)(\mathbf{x})$$

is an element of $\mathcal{B}$ for $w \in W$.

(2) *Assume that $\ell(\sigma_i w) = \ell(w) + 1$. Then we have*

$$\mathrm{val}\,(j(\sigma_i w, \mathbf{x})\,(1|_{\lambda}\sigma_i w)(\mathbf{x})) \geq \mathrm{val}(j(w, \mathbf{x})(1|_{\lambda} w)(\mathbf{x})) + \mu_i(0).$$

Proof. We prove (1) and (2) simultaneously using induction. If $w = 1$, there is nothing to prove for (1). As for (2), we obtain

$$\begin{aligned} \mathrm{val}\left(j(\sigma_i, \mathbf{x})\,(1|_\lambda \sigma_i)(\mathbf{x})\right) &= \mathrm{val}\left(x_i^{m(\alpha_i)}\left(\mathcal{P}_{0,i}(\mathbf{x}) + \mathcal{Q}_{0,i}(\mathbf{x})\mathbf{x}^{\sigma_i\cdot 0}\right)\right) \\ &= \mathrm{val}\left(x_i^{m(\alpha_i)}\left(x_i^{[\mu_i(0)]_{m(\alpha_i)}} + x_i^{\mu_i(0)-m(\alpha_i)}\right)\right) \\ &= \mathrm{val}\left(x_i^{[\mu_i(0)]_{m(\alpha_i)}+m(\alpha_i)} + x_i^{\mu_i(0)}\right) \\ &= \mu_i(0). \end{aligned}$$

Assume that $\ell(\sigma_i w) = \ell(w) + 1$. Then we have

$$\begin{aligned} & (\sigma_i w\mathbf{x})^{-\beta} j(\sigma_i w, \mathbf{x})(\mathbf{x}^\beta|_\lambda \sigma_i w)(\mathbf{x}) \\ = {}& (\sigma_i w\mathbf{x})^{-\beta} j(\sigma_i, w\mathbf{x}) j(w, \mathbf{x}) \left[\left(\mathcal{P}_{\beta,i}(\mathbf{x})\mathbf{x}^\beta + \mathcal{Q}_{\beta,i}(\mathbf{x})\mathbf{x}^{\sigma_i\cdot\beta}\right)|_\lambda w\right] \\ = {}& -(\sigma_i w\mathbf{x})^{-\beta} q^{m(\alpha_i)}(w\mathbf{x})^{m(\alpha_i)\alpha_i} j(w,\mathbf{x}) \left[\mathcal{P}_{\beta,i}(w\mathbf{x})(\mathbf{x}^\beta|_\lambda w) + \mathcal{Q}_{\beta,i}(w\mathbf{x})(\mathbf{x}^{\sigma_i\cdot\beta}|_\lambda w)\right]. \end{aligned}$$

We first consider the term having $\mathcal{P}$ factor. By induction, we need only to consider

$$\begin{aligned} & (\sigma_i w\mathbf{x})^{-\beta}(w\mathbf{x})^{m(\alpha_i)\alpha_i}(w\mathbf{x})^{[\mu_i(\beta)]_{m(\alpha_i)}\alpha_i}(w\mathbf{x})^\beta \\ = {}& q^{\beta(h_i)}(w\mathbf{x})^{-\sigma_i\beta + m(\alpha_i)\alpha_i + [\mu_i(\beta)]_{m(\alpha_i)}\alpha_i + \beta} \\ = {}& q^{\beta(h_i)}(w\mathbf{x})^{\beta(h_i)\alpha_i + m(\alpha_i)\alpha_i + [\mu_i(0)-\beta(h_i)]_{m(\alpha_i)}\alpha_i}. \end{aligned}$$

We see that

$$\begin{aligned} & \beta(h_i) + m(\alpha_i) + [\mu_i(0) - \beta(h_i)]_{m(\alpha_i)} \\ = {}& m(\alpha_i) + [\mu_i(0) - \beta(h_i)]_{m(\alpha_i)} - (\mu_i(0) - \beta(h_i)) + \mu_i(0) > \mu_i(0). \end{aligned}$$

Since $w^{-1}\alpha_i > 0$ by (1.3), we have proved the part (1). In the case $\beta = 0$, it also shows that the term having $\mathcal{P}$ factor satisfies the inequality in the part (2).

Now we consider the term having $\mathcal{Q}$ factor. Again by induction, we need only to consider

$$(\sigma_i w\mathbf{x})^{-\beta}(w\mathbf{x})^{m(\alpha_i)\alpha_i}(w\mathbf{x})^{-m(\alpha_i)\alpha_i}(w\mathbf{x})^{\sigma_i\cdot\beta} = q^{\beta(h_i)}(w\mathbf{x})^{-\sigma_i\beta+\sigma_i\cdot\beta} = q^{\beta(h_i)}(w\mathbf{x})^{\mu_i(0)\alpha_i}.$$

Therefore, (1) and (2) are true for this term as well. ∎

Let $\lambda \in P_+$. Then the sum

$$\sum_{w\in W} j(w,\mathbf{x})(1|_\lambda w)(\mathbf{x})$$

is an element of $\mathcal{B}$ by Lemma 2.8. Note that $\Delta(\mathbf{x})$ is a unit in $\mathcal{B}$. We define

$$h(\mathbf{x};\lambda) = \Delta(\mathbf{x})^{-1} \sum_{w\in W} j(w,\mathbf{x})(1|_\lambda w)(\mathbf{x}) \in \mathcal{B}$$

and

$$N(\mathbf{x};\lambda) = h(\mathbf{x};\lambda)D(\mathbf{x}) \in \mathcal{B}.$$

Assume that $f = \sum_{i=1}^\infty f_i g_i \in \widetilde{\mathcal{B}}$ with $f_i \in \widetilde{\mathcal{B}}_0$ and $g_i \in \widetilde{\mathcal{A}}$. For $w \in W$, we suppose that $f_i(w\mathbf{x}) \in \widetilde{\mathcal{B}}_0$ for all i. Then we define

$$(f|_\lambda w)(\mathbf{x}) = \sum_{i=1}^{\infty} f_i(w\mathbf{x})(g_i|_\lambda w)(\mathbf{x})$$

whenever the sum in the right yields an element of $\widetilde{\mathcal{B}}$.

Proposition 2.8.

(1) $h|_\lambda w = h$ *for all* $w \in W$.
(2) $h(0,\cdots,0,x_i,0,\cdots,0;0) = \dfrac{1+\gamma(b_i)qx_i}{1-q^{m-1}x_i^m}$.

Proof. (1) We have

$$h(\mathbf{x};\lambda) = \sum_{w\in W} \Delta(w\mathbf{x})^{-1}(1|_\lambda w)(\mathbf{x}).$$

For $u \in W$, we have $\Delta(wu\mathbf{x})^{-1} \in \widetilde{\mathcal{B}}_0$ by Lemma 2.7 (1) and

$$\sum_{w\in W} \Delta(wu\mathbf{x})^{-1}(1|_\lambda wu)(\mathbf{x}) = h(\mathbf{x};\lambda).$$

Thus $h|_\lambda w = h$ for all $w \in W$.

(2) It is straightforward. (See the equation (3.26) in [9].)

□

Let $m = m(\alpha_i)$. We write $N(\mathbf{x};\lambda) = \sum_{\mu\in Q} a_\mu \mathbf{x}^\mu$. Given any $\beta \in Q$ and a simple root α_i, we define

$$S_{\beta,i} = \{\beta + km\alpha_i \mid k \in \mathbb{Z}\},$$

and define

$$N_{\beta,i}(\mathbf{x}) = \sum_{\mu \in S_{\beta,i}} a_\mu \mathbf{x}^\mu \in \mathcal{B}.$$

Now choose $\beta \in Q$ and define

$$f_{\beta,i}(\mathbf{x}) = \begin{cases} \dfrac{N_{\beta,i}(\mathbf{x}) - \gamma(-b_i\mu_i(\beta))(q\mathbf{x}^{\alpha_i})^{(-\mu_i(\beta))_m} N_{\sigma\cdot\beta,i}(\mathbf{x})}{1 - q^{m-1}\mathbf{x}^{m\alpha_i}} & \text{if } m \nmid \mu_i(\beta); \\ \dfrac{N_{\beta,i}(\mathbf{x})}{1 - q^{m-1}\mathbf{x}^{m\alpha_i}} & \text{otherwise.} \end{cases}$$

Theorem 2.9. *We have*

$$\frac{f_{\beta,i}(\mathbf{x})}{f_{\beta,i}(\sigma_i\mathbf{x})} = \begin{cases} (q\mathbf{x}^{\alpha_i})^{\mu_i(0)-(\mu_i(\beta))_m} & \text{if } m \nmid \mu_i(\beta); \\ (q\mathbf{x}^{\alpha_i})^{\mu_i(0)-m} & \text{otherwise.} \end{cases}$$

Proof. Assume that $m \nmid \mu_i(\beta)$. We write $N_{\beta,i}(\mathbf{x}) = \left(\sum_{k\in\mathbb{Z}} a_{\beta+km\alpha_i}\mathbf{x}^{km\alpha_i}\right)\mathbf{x}^\beta$ and define

$$B_{\beta,i}(\mathbf{x}) = \sum_{k\in\mathbb{Z}} a_{\beta+km\alpha_i}\mathbf{x}^{km\alpha_i} \in \mathcal{B}_0,$$

so that we have $N_{\beta,i}(\mathbf{x}) = B_{\beta,i}(\mathbf{x})\mathbf{x}^\beta$. Define

$$F_{\beta,i}(\mathbf{x}) = \frac{N_{\beta,i}(\mathbf{x}) + N_{\sigma_i\cdot\beta,i}(\mathbf{x})}{1 - q^{m-1}\mathbf{x}^{m\alpha_i}} = \frac{B_{\beta,i}(\mathbf{x})\mathbf{x}^\beta + B_{\sigma_i\cdot\beta,i}(\mathbf{x})\mathbf{x}^{\sigma_i\cdot\beta}}{1 - q^{m-1}\mathbf{x}^{m\alpha_i}}.$$

Then $F_{\beta,i}$ is invariant under $|_\lambda\sigma_i$ by Proposition 2.9 (1). Applying σ_i to $F_{\beta,i}$, we obtain

$$F_{\beta,i}(\mathbf{x}) = (F_{\beta,i}|_\lambda\sigma_i)(\mathbf{x}) = \frac{B_{\beta,i}(\sigma_i\mathbf{x})(\mathbf{x}^\beta|_\lambda\sigma_i)(\mathbf{x}) + B_{\sigma_i\cdot\beta,i}(\sigma_i\mathbf{x})(x^{\sigma_i\cdot\beta}|_\lambda\sigma_i)(\mathbf{x})}{1 - q^{-m-1}\mathbf{x}^{-m\alpha_i}}.$$

Using this, we compute further and obtain

$$f_{\beta,i}(\mathbf{x}) = \frac{B_{\beta,i}(\sigma_i\mathbf{x})(q\mathbf{x}^{\alpha_i})^{[\mu_i(\beta)]_m}\mathbf{x}^\beta - B_{\sigma_i\cdot\beta,i}(\sigma_i\mathbf{x})\gamma(-b_i\mu_i(\beta))(q\mathbf{x}^{\alpha_i})^{-m-\mu_i(\beta)}\mathbf{x}^{\sigma_i\cdot\beta}}{1 - q^{-m-1}\mathbf{x}^{-m\alpha_i}}.$$

Now the assertion of the theorem follows from this.

The proof in the case that $m|\mu_i(\beta)$ is similar, and we omit the details. □

3. The Coefficients H

We specialize $\gamma(i)$ to be

$$\gamma(i) = \begin{cases} g(1,\varpi;i)/q & \text{if } i \not\equiv 0 (\text{mod } n); \\ -1 & \text{otherwise,} \end{cases}$$

where ϖ is a prime in $\mathfrak{o}_S$. We define

$$\varpi_Q^\beta = (\varpi^{k_1}, \cdots, \varpi^{k_r}) \in (\mathfrak{o}_S)^r \quad \text{and} \quad \varpi_P^\lambda = (\varpi^{l_1}, \cdots, \varpi^{l_r}) \in (\mathfrak{o}_S)^r$$

where $\beta = \sum_{i=1}^r k_i \alpha_i \in Q_+$ and $\lambda = \sum_{i=1}^r l_i \omega_i \in P_+$. For $\mathbf{m} = (m_1, \cdots, m_r) \in (\mathfrak{o}_S)^r$, we have decompositions

$$\mathbf{m} = \prod_{\varpi:\text{ prime}} \varpi_Q^{\beta_\varpi} = \prod_{\varpi:\text{ prime}} \varpi_P^{\lambda_\varpi} \tag{3.1}$$

for $\beta_\varpi \in Q_+$ and $\lambda_\varpi \in P_+$ for each prime ϖ. Denote the $\mathbf{x}^\beta$-coefficient of $N(\mathbf{x};\lambda)$ by

$$H(\varpi_Q^\beta; \varpi_P^\lambda). \tag{3.2}$$

If we have $\gcd(c_1 \cdots c_r, c_1' \cdots c_r') = 1$ for $\mathbf{c} = (c_1, \cdots, c_r) \in (\mathfrak{o}_S)^r$ and $\mathbf{c}' = (c_1', \cdots, c_r') \in (\mathfrak{o}_S)^r$, we require that the twisted multiplicativity should hold:

$$H(\mathbf{c}\mathbf{c}'; \mathbf{m}) = \xi_B(\mathbf{c}, \mathbf{c}') H(\mathbf{c}; \mathbf{m}) H(\mathbf{c}'; \mathbf{m}). \tag{3.3}$$

We also require the relation

$$H(\mathbf{c}; \mathbf{m}\mathbf{m}') = \left[\frac{\mathbf{m}'}{\mathbf{c}}\right]^{-B} H(\mathbf{c}; \mathbf{m}) \tag{3.4}$$

if $\gcd(c_1 \cdots c_r, m_1' \cdots m_r') = 1$ for $\mathbf{c} = (c_1, \cdots, c_r) \in (\mathfrak{o}_S)^r$ and $\mathbf{m}' = (m_1', \cdots, m_r') \in (\mathfrak{o}_S)^r$. Combining (3.2), (3.3) and (3.4), we have defined the coefficients $H(\mathbf{c}; \mathbf{m})$ for any $\mathbf{c}, \mathbf{m} \in (\mathfrak{o}_S)^r$.

Lemma 3.5. *Assume that $\beta \in Q_+$ and $\lambda \in P_+$. There exist constants $c_A, M_A \in \mathbb{R}_{>0}$ such that*

$$|H(\varpi_Q^\beta; \varpi_P^\lambda)| < M_A |\varpi|^{c_A d(\beta)}$$

for all primes $\varpi \in \mathfrak{o}_S$.

Given $\beta \in Q_+$, we define

$$N^{(\varpi)}_{\beta,i}(\mathbf{x};\mathbf{m}) = \sum_{j\geq 0} H(\varpi_Q^{\beta+jm\alpha_i};\mathbf{m})\mathbf{x}^{\beta+jm\alpha_i},$$

where $m = m(\alpha_i)$. Let $\lambda \in P_+$ be such that ϖ_P^λ is the ϖ-factor in the decomposition of $\mathbf{m}$, i.e. $\lambda = \lambda_\varpi$ in (3.1) . We write

$$\mathbf{m}' = \mathbf{m}/\varpi_P^\lambda = (m_1', \cdots, m_r'),$$

and set $\mu_i(\beta) = \mu_{i,\lambda}(\beta) = (\lambda + \rho - \beta)(h_i)$ as before. We put $q = |\varpi|$ and define

$$f^{(\varpi)}_{\beta,i}(\mathbf{x};\mathbf{m}) = \begin{cases} \dfrac{N^{(\varpi)}_{\beta,i}(\mathbf{x};\mathbf{m}) - q^{-1}g(m_i',\varpi;-b_i\mu_i(\beta))(q\mathbf{x}^{\alpha_i})^{(-\mu_i(\beta))_m}N^{(\varpi)}_{\sigma_i\cdot\beta,i}(\mathbf{x};\mathbf{m})}{1-q^{m-1}\mathbf{x}^{m\alpha_i}} & \text{if } \mu_i(\beta)\nmid m; \\ \dfrac{N^{(\varpi)}_{\beta,i}(\mathbf{x};\mathbf{m})}{1-q^{m-1}\mathbf{x}^{m\alpha_i}} & \text{otherwise .} \end{cases}$$

Lemma 3.6. *We have*

$$N^{(\varpi)}_{\beta,i}(\mathbf{x};\mathbf{m}) = \left[\frac{\mathbf{m}'}{\varpi_Q^\beta}\right]^{-B} N_{\beta,i}(\mathbf{x}) \quad \text{and} \quad f^{(\varpi)}_{\beta,i}(\mathbf{x};\mathbf{m}) = \left[\frac{\mathbf{m}'}{\varpi_Q^\beta}\right]^{-B} f_{\beta,i}(\mathbf{x}).$$

Proof. From the twisted multiplicativity, we obtain

$$\begin{aligned} H(\varpi_Q^{\beta+jm\alpha_i};\mathbf{m}) &= H(\varpi_Q^{\beta+jm\alpha_i};\varpi_P^\lambda\mathbf{m}') \\ &= \left[\frac{\mathbf{m}'}{\varpi_Q^{\beta+jm\alpha_i}}\right]^{-B} H(\varpi_Q^{\beta+jm\alpha_i};\varpi_P^\lambda) \\ &= \left[\frac{\mathbf{m}'}{\varpi_Q^{\beta}}\right]^{-B} H(\varpi_Q^{\beta+jm\alpha_i};\varpi_P^\lambda), \end{aligned}$$

since $\left(\dfrac{m_i'}{\varpi^{jm}}\right)^{-b_i} = \left(\dfrac{m_i'}{\varpi^{j}}\right)^{-b_i m} = 1$. Then we have

$$N^{(\varpi)}_{\beta,i}(\mathbf{x};\mathbf{m}) = \left[\frac{\mathbf{m}'}{\varpi_Q^\beta}\right]^{-B} \sum_{j\geq 0} H(\varpi_Q^{\beta+jm\alpha_i};\varpi_P^\lambda)\mathbf{x}^{\beta+jm\alpha_i} = \left[\frac{\mathbf{m}'}{\varpi_Q^\beta}\right]^{-B} N_{\beta,i}(\mathbf{x}).$$

Since we have $\sigma_i\cdot\beta = \beta + \mu_i(\beta)\alpha_i$ and the multiplicativitiy of the power residue symbol, we obtain

$$N^{(\varpi)}_{\sigma_i\cdot\beta,i}(\mathbf{x};\mathbf{m}) = \left[\frac{\mathbf{m}'}{\varpi_Q^{\sigma_i\cdot\beta}}\right]^{-B} N_{\sigma_i\cdot\beta,i}(\mathbf{x}) = \left[\frac{\mathbf{m}'}{\varpi_Q^\beta}\right]^{-B} \left(\frac{m_i'}{\varpi}\right)^{-b_i\mu_i(\beta)} N_{\sigma_i\cdot\beta,i}(\mathbf{x}).$$

On the other hand,

$$g(m_i', \varpi; -b_i\mu_i(\beta)) = \left(\frac{m_i'}{\varpi}\right)^{b_i\mu_i(\beta)} g(1, \varpi; -b_i\mu_i(\beta)).$$

Now it it straightforward to see that

$$f_{\beta,i}^{(\varpi)}(\mathbf{x};\mathbf{m}) = \left[\frac{\mathbf{m}'}{\varpi_Q^{\beta}}\right]^{-B} f_{\beta,i}(\mathbf{x}).$$

□

Theorem 3.7. *We have*

$$\frac{f_{\beta,i}^{(\varpi)}(\mathbf{x};\mathbf{m})}{f_{\beta,i}^{(\varpi)}(\sigma_i\mathbf{x};\mathbf{m})} = \begin{cases} (q\mathbf{x}^{\alpha_i})^{\mu_i(0)-(\mu_i(\beta))_m} & \text{if } m \nmid \mu_i(\beta); \\ (q\mathbf{x}^{\alpha_i})^{\mu_i(0)-m} & \text{otherwise.} \end{cases}$$

Proof. The assertion follows from Lemma 3.6 and Theorem 2.10. □

4. Rank One Computations

For $j \in \mathbb{Z}_{>0}$, $\Psi \in \mathcal{M}_j(\Omega)$ and $a \in \mathfrak{o}_S$, we define

$$\mathcal{D}(s, a; \Psi, j) = \sum_{0 \neq c \in \mathfrak{o}_S/\mathfrak{o}_S^{\times}} g(a, c; j)\Psi(c)|c|^{-s}|a|^{s/2}.$$

Let $m = n/\gcd(n, j)$ and set

$$G_m(s) = \Big((2\pi)^{-(m-1)(s-1)}\Gamma(ms-m)/\Gamma(s-1)\Big)^{[F:\mathbb{Q}]/2}.$$

Define

$$\mathcal{D}^*(s, a; \Psi, j) = G_m(s)\zeta_F(ms-m+1)\mathcal{D}(s, a, \Psi, j),$$

where ζ_F is the Dedekind zeta function of F. If $\Psi \in \mathcal{M}_j(\Omega)$ and $\eta \in F_S^{\times}$ we define

$$\hat{\Psi}_\eta(c) = (\eta, c)_S^j \Psi(\eta c) \quad \text{and} \quad \tilde{\Psi}_\eta(c) = (\eta, c)_S^j \Psi(\eta^{-1}c^{-1}).$$

Theorem 4.1. [1] *The function $\mathcal{D}^*(s,a;\Psi,j)$ has a meromorphic continuation to $\mathbb{C}$ and is holomorphic except for possible simple poles at $s = 1 \pm 1/m$. Moreover, there exist S-Dirichlet polynomials $P(s;a\eta,j)$ such that*

$$\mathcal{D}^*(s,a;\Psi,j) = \sum_{\eta \in F_S^\times / F_S^{\times,n}} P(s;a\eta,j)\mathcal{D}^*(2-s,a;\tilde{\Psi}_\eta,j).$$

Let $\mathbf{m} = (m_1, \cdots, m_r) \in (\mathfrak{o}_S)^r$ be fixed for the rest of this section. Let $\mathfrak{A}$ be the ring of Laurent polynomials in $|\varpi_v|^{s_i}$, $i = 1, ..., r$, where v runs over the places in S_{fin}. We define

$$\mathfrak{M}_B(\Omega) = \mathfrak{A} \otimes \mathcal{M}_B(\Omega) \quad \text{and} \quad \mathfrak{M}_j(\Omega) = \mathfrak{A} \otimes \mathcal{M}_j(\Omega).$$

We write

$$s_i = \alpha_i(\mathbf{s}) \qquad \text{for } \mathbf{s} \in \mathfrak{h}$$

and regard an element of $\mathfrak{M}_B(\Omega)$ as a function on $\mathfrak{h} \times (F_S^\times)^r$. Denote by ι the diagonal embedding:

$$\iota : F_S^\times \to (F_S^\times)^r, \qquad x \mapsto (x, x, \cdots, x).$$

If $\Psi \in \mathfrak{M}_B(\Omega)$ and $\mathbf{a} = (a_1, \cdots, a_r) \in (\mathfrak{o}_S / \mathfrak{o}_S^\times)^r$, we define

$$\Psi_i^{\mathbf{a}}(\mathbf{s};c) = (\mathbf{a}, \iota(c))_{S,i}^B \; \Psi(\mathbf{s}; a_1, \cdots, a_i c, \cdots, a_r).$$

Lemma 4.2. [3] *We have*

$$\Psi_i^{\mathbf{a}} \in \mathfrak{M}_{b_i}(\Omega).$$

We define a shifted action of W on $\mathfrak{h}$ by

$$\sigma_i \circ \mathbf{s} = \sigma_i(\mathbf{s} - \rho^\vee) + \rho^\vee.$$

Now we define an action of σ_i on $\mathfrak{M}_B(\Omega)$ as follows. For $\Psi \in \mathfrak{M}_B(\Omega)$, we set

$$(\sigma_i \Psi)(\mathbf{s};\mathbf{a}) = \sum_{\eta \in F_S^\times / F_S^{\times,n}} (\iota(\eta), \mathbf{a})_{S,i}^B \; P(s_i\,;\, \eta m_i \mathbf{a}^{-h_i}, b_i) \; \Psi(\sigma_i \circ \mathbf{s}; a_1, \cdots, a_i \eta^{-1}, \cdots, a_r)$$

where

$$\mathbf{a}^{-h_i} = \prod_j a_j^{-\alpha_j(h_i)} = \prod_j a_j^{-a_{ij}}.$$

Warning: Remember that $b_i = (\alpha_i, \alpha_i)$ in this paper. The element $\mathbf{a}^{-h_i}$ is denoted by b_i in [9].

Proposition 4.3. [3] *If $\Psi \in \mathfrak{M}_B(\Omega)$ then $\sigma_i \Psi \in \mathfrak{M}_B(\Omega)$.*

Let $\hat{\mathbf{a}}$ be the $(r-1)$-tuple $(a_1, \cdots, \hat{a}_i, \cdots, a_r)$ where the hat on a_i indicates that this entry is omitted. Similarly, we let $\hat{\mathbf{m}} = (m_1, \cdots, \hat{m}_i, \cdots, m_r)$. Let $\Psi \in \mathfrak{M}_B(\Omega)$. We define

$$\mathcal{E}(\mathbf{s}, \hat{\mathbf{a}}; \mathbf{m}, \Psi, i) = \sum_{0 \neq a_i \in \mathfrak{o}_S/\mathfrak{o}_S^\times} H(a_1, \cdots, a_i, \cdots, a_r; \mathbf{m}) \Psi(a_1, \cdots, a_i, \cdots, a_r) |a_i|^{-\alpha_i(\mathbf{s})} |m_i|^{\omega_i(\mathbf{s})}.$$

Let $m = m(\alpha_i)$ and $s_i = \alpha_i(\mathbf{s})$. We define

$$\mathcal{E}^*(\mathbf{s}, \hat{\mathbf{a}}; \mathbf{m}, \Psi, i) = G_m(s_i) \zeta_F(ms_i - m + 1) \mathcal{E}(\mathbf{s}, \hat{\mathbf{a}}; \mathbf{m}, \Psi, i).$$

Proposition 4.4. [9] *We have*

$$\mathcal{E}^*(\mathbf{s}, \hat{\mathbf{a}}; \mathbf{m}, \Psi, i) = |\hat{\mathbf{a}}|_Q^{(s_i-1)h_i} \ \mathcal{E}^*(\sigma_i \circ \mathbf{s}, \hat{\mathbf{a}}; \mathbf{m}, \sigma_i \Psi, i),$$

where

$$|\hat{\mathbf{a}}|_Q^{(s_i-1)h_i} = \prod_{j \neq i} |a_j|^{(s_i-1)\alpha_j(h_i)} = \prod_{j \neq i} |a_i|^{(s_i-1)a_{ij}}.$$

(See the definition in (5.1).)

5. The Multiple Dirichlet Series

Let $\mathbf{m} = (m_1, \cdots, m_r) \in (\mathfrak{o}_S)^r$ be fixed. Then we have $\mathbf{m} = \prod_\varpi \varpi_P^{\lambda_\varpi}$ with $\lambda_\varpi \in P_+$ fixed. If $\mathbf{c} = (c_1, \cdots, c_r) \in (\mathfrak{o}_S)^r$ and $\mathbf{s} \in \mathfrak{h}$, we set

$$|\mathbf{c}|_Q^{\mathbf{s}} = \prod_{i=1}^r |c_i|^{\alpha_i(\mathbf{s})} = |c_1|^{s_1} \cdots |c_r|^{s_r} \quad \text{and} \quad |\mathbf{c}|_P^{\mathbf{s}} = \prod_{i=1}^r |c_i|^{\omega_i(\mathbf{s})}. \tag{5.1}$$

Let $\Psi \in \mathfrak{M}_B(\Omega)$, and we define a function $Z(\mathbf{s}; \mathbf{m}, \Psi)$ on $\mathfrak{h}$ by

$$Z(\mathbf{s}; \mathbf{m}, \Psi) = \sum_{\mathbf{c}} H(\mathbf{c}; \mathbf{m}) \Psi(\mathbf{s}; \mathbf{c}) |\mathbf{c}|_Q^{-\mathbf{s}} |\mathbf{m}|_P^{\mathbf{s}},$$

where the sum is over $\mathbf{c} = (c_1, \cdots, c_r)$ such that $0 \neq c_i \in \mathfrak{o}_S/\mathfrak{o}_S^\times$ for $i = 1, ..., r$.

Theorem 5.2. *Assume that* $\Psi \in \mathfrak{M}_B(\Omega)$*. The series* $Z(\mathbf{s}; \mathbf{m}, \Psi)$ *absolutely converges for* $\mathbf{s} \in \mathfrak{h}$ *satisfying the condition:*

$$\mathfrak{Re}(\alpha_i(\mathbf{s})) = \mathfrak{Re}(s_i) > c_A + 1 \qquad \textit{for each } i = 1, ..., r.$$

Proof. We may assume that $\mathbf{s}$ is real, i.e. $\mathbf{s} \in \mathfrak{h}_{\mathbb{R}} = \mathbb{R} \otimes P^\vee$. Suppose that

$$s_i \geq c_A + 1 + \varepsilon \qquad \text{for each } i = 1, ..., r.$$

Since the function Ψ is bounded, it is sufficient to consider

$$\sum |H(\mathbf{c};\mathbf{m})||\mathbf{c}|_Q^{-\mathbf{s}} = \prod_{\varpi} \sum_{\beta \in Q^+} |H(\varpi_Q^\beta; \varpi_P^{\lambda_\varpi})||\varpi|^{-\beta(\mathbf{s})}.$$

Since $d(\beta) = \beta(\rho^\vee)$, we obtain, by Lemma 3.5,

$$\begin{aligned} \sum_{\beta \in Q^+} |H(\varpi_Q^\beta; \varpi_P^{\lambda_\varpi})||\varpi|^{-\beta(\mathbf{s})} &\leq 1 + \sum_{\beta \in Q^+ \setminus \{0\}} M_A |\varpi|^{c_A d(\beta) - \beta(\mathbf{s})} \\ &= 1 + \sum_{\beta \in Q^+ \setminus \{0\}} M_A |\varpi|^{\beta(c_A \rho^\vee - \mathbf{s})} = 1 + O(|\varpi|^{-1-\varepsilon}). \end{aligned}$$

□

For any $\alpha \in \Phi$, we define

$$\zeta_\alpha(\mathbf{s}) = \zeta_F\left(1 + m(\alpha)\alpha(\mathbf{s} - \rho^\vee)\right),$$

and

$$G_\alpha(\mathbf{s}) = G_{m(\alpha)}\left(\frac{1}{2} + \frac{1}{2}\alpha(\mathbf{s} - \rho^\vee)\right).$$

It is easy to see that

$$G_\alpha(\sigma_i \circ \mathbf{s}) = G_{\sigma_i \alpha}(\mathbf{s}) \quad \text{and} \quad \zeta_\alpha(\sigma_i \circ \mathbf{s}) = \zeta_{\sigma_i \alpha}(\mathbf{s}).$$

In particular,

$$G_{\alpha_i}(\sigma_i \circ \mathbf{s}) = G_{-\alpha_i}(\mathbf{s}) \quad \text{and} \quad \zeta_{\alpha_i}(\sigma_i \circ \mathbf{s}) = \zeta_{-\alpha_i}(\mathbf{s}).$$

Then we define

$$G(w, \mathbf{s}) = \prod_{\alpha \in \Phi(w)} \frac{G_\alpha(\mathbf{s})\zeta_\alpha(\mathbf{s})}{G_{-\alpha}(\mathbf{s})\zeta_{-\alpha}(\mathbf{s})},$$

Theorem 5.3. *The funcion $Z(\mathbf{s};\mathbf{m},\Psi)$ satisfies the functional equation*

$$Z(w \circ \mathbf{s}; \mathbf{m}, w\Psi) = G(w, \mathbf{s}) Z(\mathbf{s}; \mathbf{m}, \Psi)$$

for each $w \in W$.

Proof. Since we have

$$\Phi(w) = \{\sigma_{i_k} \cdots \sigma_{i_{j+1}} \alpha_{i_j} \mid j = 1, ..., k\} \qquad \text{for } w = \sigma_{i_1} \cdots \sigma_{i_k},$$

the assertion follows from Lemma 4.4. □

Let $\mathfrak{h}_{\mathbb{R}} = \mathbb{R} \otimes P^\vee$. For $\mathbf{s} \in \mathfrak{h}$, we write $\mathbf{s} = \mathfrak{Re}(\mathbf{s}) + \sqrt{-1}\,\mathfrak{Im}(\mathbf{s})$ with $\mathfrak{Re}(\mathbf{s}), \mathfrak{Im}(\mathbf{s}) \in \mathfrak{h}_{\mathbb{R}}$. Let

$$\mathfrak{F} = \{\mathbf{s} \in \mathfrak{h}_{\mathbb{R}} \mid \langle \alpha_i, \mathbf{s} \rangle \geq 1 \text{ for all } i = 1, ..., r\}.$$

We define the shifted Tits cone $\mathfrak{X} \subseteq \mathfrak{h}_{\mathbb{R}}$ to be

$$\mathfrak{X} = \bigcup_{w \in W} w \circ \mathfrak{F}.$$

Proposition 5.4. [14]

(1) *$\mathfrak{X}$ is a convex cone.*

(2) *$\mathfrak{X} = \mathfrak{h}_{\mathbb{R}}$ if and only if $|W| < \infty$.*

Theorem 5.5. *The Dirichlet series $Z(\mathbf{s}; \mathbf{m}, \Psi)$ has meromorphic continuation to all $\mathbf{s} \in \mathfrak{h}$ such that $\mathfrak{Re}(\mathbf{s}) \in \mathfrak{X}$.*

Proof. We use the standard argument. □

References

[1] B. Brubaker and D. Bump, *On Kubota's Dirichlet series*, J. Reine Angew. Math. **598** (2006), 159–184.

[2] B. Brubaker, D. Bump, G. Chinta, S. Friedberg and J. Hoffstein, *Weyl group multiple Dirichlet series. I*, Multiple Dirichlet series, automorphic forms, and analytic number theory, 91–114, Proc. Sympos. Pure Math., **75**, Amer. Math. Soc., Providence, RI, 2006.

[3] B. Brubaker, D. Bump and S. Friedberg, *Weyl group multiple Dirichlet series. II. The stable case*, Invent. Math. **165** (2006), no. 2, 325–355.

[4] ———, *Twisted Weyl group multiple Dirichlet series: the stable case*, Eisenstein series and applications, 1–26, Progr. Math., **258**, Birkhuser Boston, Boston, MA, 2008.

[5] ———, *Weyl group multiple Dirichlet series, Eisenstein series and crystal bases*, preprint, 2008.

[6] B. Brubaker, D. Bump, S. Friedberg and J. Hoffstein, *Weyl group multiple Dirichlet series. III. Eisenstein series and twisted unstable A_r*, Ann. of Math. (2) **166** (2007), no. 1, 293–316.

[7] I. Cherednik, *Double affine Hecke algebras and Macdonald's conjectures*, Ann. of Math. **141** (1995), 191–216.

[8] G. Chinta and P. Gunnells, *Weyl group multiple Dirichlet series constructed from quadratic characters*, Invent. Math. 167 (2007), no. 2, 327–353.

[9] ———, *Constructing Weyl group multiple Dirichlet series*, to Appear in J. AMS.

[10] J. Hong and S.-J. Kang, *Introduction to quantum groups and crystal bases*, Graduate Studies in Mathematics **42**, American Mathematical Society, Providence, RI, 2002.

[11] H. Garland, *Eisenstein series on arithmetic quotients of loop groups*, Math. Res. Lett. **6** (1999), no. 5-6, 723–733.

[12] ———, *Certain Eisenstein series on loop groups: convergence and the constant term*, Algebraic groups and arithmetic, 275–319, Tata Inst. Fund. Res., Mumbai, 2004.

[13] ———, *Absolute convergence of Eisenstein series on loop groups*, Duke Math. J. **135** (2006), no. 2, 203–260.

[14] V.G. Kac, *Infinite-dimensional Lie algebras*, Third edition, Cambridge University Press, Cambridge, 1990.

[15] S.-J. Kang, *Crystal bases for quantum affine algebras and combinatorics of Young walls*, Proc. London Math. Soc. (3) **86** (2003), no. 1, 29–69.

[16] J. Neukirch, *Algebraic number theory*, Grundlehren der Mathematischen Wissenschaften, **322**, Springer-Verlag, Berlin, 1999.

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