1. Can you use the proof of Pfister’s theorem to obtain an identity
\[(a^2 + b^2)(c^2 + d^2) = (?)^2 + (?)^2\]
which is not the usual identity?

2. What formula does Pfister’s 4-square identity provide for the product
\[(x^2_1 + x^2_2 + x^2_3)(y^2_1 + y^2_2 + y^2_3)\]
when we view the factors as 4-square sums with \(x_4 = 0\) and \(y_4 = 0\)?

3. Let’s look at sums of two squares in \(\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\}\).
   a) For \(\alpha = a + b\sqrt{2}\) in \(\mathbb{Z}[\sqrt{2}]\), set \(\overline{\alpha} = a - b\sqrt{2}\). Show \(\overline{\alpha} + \overline{\beta} = \overline{\alpha + \beta}\) and \(\overline{\alpha\beta} = \overline{\alpha}\overline{\beta}\).
   b) If \(\alpha \in \mathbb{Z}[\sqrt{2}]\) is a sum of squares, show \(\overline{\alpha}\) is as well.
   c) Show \(1 + \sqrt{2}\) is not a sum of squares in \(\mathbb{Z}[\sqrt{2}]\), even though \(1 + \sqrt{2}\) is positive.
   d) Show \(3 + \sqrt{2}\) is not a sum of squares in \(\mathbb{Z}[\sqrt{2}]\), even though \(3 + \sqrt{2}\) and \(3 - \sqrt{2}\) are positive.
   e) Show \(3 + \sqrt{2}\) is a sum of squares in \(\mathbb{Q}[\sqrt{2}]\). What about \(1 + \sqrt{2}\)?
   f) Try other examples in \(\mathbb{Q}[\sqrt{2}]\). When do you think an element of \(\mathbb{Q}[\sqrt{2}]\) is a sum of two squares in \(\mathbb{Q}[\sqrt{2}]\)?

4. Is every element of \(\mathbb{Q}[i]\) a sum of squares in \(\mathbb{Q}[i]\)?

5. The “fifteen theorem” says \(a_1 x^2_1 + \cdots + a_r x^2_r\), with \(a_i \in \mathbb{Z}^+\), represents all positive integers using \(x_i \in \mathbb{Z}\) provided it represents the integers from 1 to 15. (The case \(a = 1\) recovers Lagrange’s 4-square theorem.) Use the fifteen theorem to decide which of the polynomials \(x^2 + y^2 + z^2 + aw^2\), for \(2 \leq a \leq 10\), represent all positive integers.