Math 3231 - Abstract Algebra II — Due in MSB 318

Problem Set 5 — 4/10/09 at 3 PM

Reading: text §§8.3 and 8.5 (through Theorem 8.5.4), handout on cyclotomic extensions.
Note: The **final exam** is scheduled for May 6 (Wednesday) 3:30-5:30 in MSB 415.

> *Since the beginning of the [19th] century, computational procedures have become so complicated that any progress by those means has become impossible, without the elegance which modern mathematicians have brought to bear on their research, and by means of which the spirit comprehends quickly and in one step a great many computations. [...] Classify [operations] according to their complexities rather than their appearances! This, I believe, is the mission of future mathematicians. This is the road on which I am embarking in this work.*
>
> E. Galois

1. Use the isomorphism $\mathrm{Gal}(\mathbf{Q}(\zeta_{11})/\mathbf{Q}) \cong (\mathbf{Z}/(11))^\times$ to count the subfields of $\mathbf{Q}(\zeta_{11})$ and to find a generating element for each one. (Start by determining the subgroups of the *cyclic* group $(\mathbf{Z}/(11))^\times$.)

2. Let $L = \mathbf{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$.

   a) Show $[L : \mathbf{Q}] = 8$ and give a $\mathbf{Q}$-basis of L. (Hint: Use the list of subfields of $\mathbf{Q}(\sqrt{2}, \sqrt{3})$ to see what the degree of $\sqrt{5}$ is over $\mathbf{Q}(\sqrt{2}, \sqrt{3})$.)

   b) Show $\mathrm{Gal}(L/\mathbf{Q}) \cong \{\pm 1\}^3$. (Hint: Think about the effect any $\sigma \in \mathrm{Gal}(L/\mathbf{Q})$ has on $\sqrt{2}, \sqrt{3}$, and $\sqrt{5}$.)

   c) Compute the Galois orbit of $\sqrt{2} + \sqrt{3} + \sqrt{5}$; is this sum a generator of $\mathbf{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})/\mathbf{Q}$?

3. The polynomial $f(T) = T^4 + T + 2$ is irreducible in $\mathbf{F}_3[T]$. Set $\mathbf{F}_{81} = \mathbf{F}_3(\alpha)$, where $f(\alpha) = 0$.

   a) Express the powers α^k for $0 \leq k \leq 9$ in terms of the $\mathbf{F}_3$-basis $\{1, \alpha, \alpha^2, \alpha^3\}$. (Be sure all answers here are correct; they are needed in part b.)

   b) Use iterates of the 3rd power map on $\mathbf{F}_{81}$ to determine the degree over $\mathbf{F}_3$ of the following elements:
   $$\alpha^2, \quad \alpha^3 + \alpha, \quad \alpha^3 + \alpha^2 + 2\alpha.$$

4. Let ζ be a primitive 7th root of unity in a field extension of $\mathbf{F}_2$.

   a) Use Galois theory for finite fields to show $[\mathbf{F}_2(\zeta) : \mathbf{F}_2] = 3$. What does this tell you about the irreducible factors of $\Phi_7(T)$ in $\mathbf{F}_2[T]$?

   b) Give an explicit irreducible factorization of $\Phi_7(T)$ in $\mathbf{F}_2[T]$.