Math 3231 - Abstract Algebra II
Problem Set 4

Due in MSB 318
3/27/09 at 3 PM

Reading: §§6.4, 8.1, 8.2.

It is the constant aim of the mathematician to reduce all his expressions to their lowest terms, to entrench every superfluous word and phrase, and to condense the Maximum of meaning into the Minimum of Language. J. J. Sylvester

1. Let R be a commutative ring. For $f(T) = c_0 + c_1T + c_2T^2 + \cdots + c_nT^n = \sum_{i=0}^{n} c_iT^i$ in $R[T]$, define its derivative to be

$$f'(T) = c_1 + 2c_2T + \cdots + nc_nT^{n-1} = \sum_{i=0}^{n} ic_iT^{i-1} \in R[T].$$

(In particular, constant polynomials have derivative 0: the term at $i = 0$ is 0, so no negative exponents occur.) Notice we are *not* using limits. This formula for the derivative is the *definition*. Of course it is inspired by results from calculus, where the derivative is defined with limits, but here we work purely algebraically.

Prove the following properties of the derivative on $R[T]$. (No limits to be used, just algebra.)

(a) $(f(T) + g(T))' = f'(T) + g'(T)$ and $(cf(T))' = cf'(T)$ for all $f(T)$ and $g(T)$ in $R[T]$ and $c \in R$.

(b) $(f(T)g(T))' = f(T)g'(T) + f'(T)g(T)$ for all $f(T)$ and $g(T)$ in $R[T]$.

(c) $(f(T)^m)' = mf(T)^{m-1}f'(T)$ for all $f(T)$ in $R[T]$ and $m \geq 1$.

(d) $(f(g(T)))' = f'(g(T))g'(T)$ for all $f(T)$ and $g(T)$ in $R[T]$.

(Hint on parts b and d: first check in the case when $f(T) = aT^m$ is a monomial and $g(T)$ is arbitrary. Then use additivity to extend this to the general case, using part a.)

2. Determine the degree of the splitting field of $T^4 + 1$ over the following fields: $\mathbf{Q}$, $\mathbf{F}_2$, and $\mathbf{F}_3$. (First determine the irreducible factorization of $T^4 + 1$ over each field.)

3. If $a \in \mathbf{F}_p^{\times}$, use polynomial derivatives and gcd's to show $T^n - a \in \mathbf{F}_p[T]$ is separable as long as p is not a factor of n. What can you say if p is a factor of n?

4. In $\mathrm{Gal}(\mathbf{Q}(\sqrt{2}, \sqrt{3})/\mathbf{Q})$, let σ_1 and σ_2 be determined by $\sigma_1(\sqrt{2}) = -\sqrt{2}$, $\sigma_1(\sqrt{3}) = \sqrt{3}$ and $\sigma_2(\sqrt{2}) = \sqrt{2}$, $\sigma_2(\sqrt{3}) = -\sqrt{3}$. Set $\sigma_3 = \sigma_1 \circ \sigma_2$.

Show σ_3^2 is the identity on $\mathbf{Q}(\sqrt{2}, \sqrt{3})$ and $\{x \in \mathbf{Q}(\sqrt{2}, \sqrt{3}) : \sigma_3(x) = x\} = \mathbf{Q}(\sqrt{6})$. (Hint: Write x as a $\mathbf{Q}$-linear combination of $\{1, \sqrt{2}, \sqrt{3}, \sqrt{6}\}$ and then compare x and $\sigma_3(x)$.)