Math 3231 - Abstract Algebra II
Problem Set 3

Due in MSB 318
2/27/09 at 3 PM

Reading: §§5.3, 6.1, 6.2, handout on dimensions. (Optional: 5.4)
Note: The **midterm** exam is in class on March 5th.

> *It is the common curse of all general and abstract theories that they have to be far advanced before yielding useful results in concrete problems.* H. Weyl

1. Let $F = \mathbf{Q}[\sqrt[3]{2}]$. Every element in F has the (unique!) form $a + b\sqrt[3]{2} + c\sqrt[3]{4}$ for some rational a, b, and c. Determine the multiplicative inverse of $3 + \sqrt[3]{2} + \sqrt[3]{4}$ in two ways:

   a) Turn the condition
   $$(3 + \sqrt[3]{2} + \sqrt[3]{4})(a + b\sqrt[3]{2} + c\sqrt[3]{4}) = 1$$
   into a system of 3 linear equations in 3 unknowns a, b, c and solve it by methods of linear algebra.

   b) Use the Euclidean algorithm and back-substitution on polynomials to solve
   $$(T^2 + T + 3)u(T) + (T^3 - 2)v(T) = 1$$
   for $u(T)$ and $v(T)$ in $\mathbf{Q}[T]$. Then substitute $\sqrt[3]{2}$ for T to get an equation in $\mathbf{Q}[\sqrt[3]{2}]$.

2. For each of the following numbers, find (with justification) its minimal polynomial in $\mathbf{Q}[T]$: $4 + \sqrt{3}$, $\frac{5}{\sqrt[3]{2}}$, $\sqrt{5} + \sqrt{7}$.

3. Let F be a field containing $\mathbf{F}_2$ and define $\varphi\colon F \to F$ to be the squaring function on F: $\varphi(x) = x^2$. This is a ring homomorphism since squaring is additive in a field where $2 = 0$.

   a) For any $f(T) = a_0 + a_1T + \cdots + a_nT^n$ in $\mathbf{F}_2[T]$ and $x \in F$, show $\varphi(f(x)) = f(x^2)$. (In particular, if $f(x) = 0$ then $f(x^2) = 0$, so we get a new root from an old root x by squaring!)

   b) In the field $\mathbf{F}_2[\alpha]$, where $\alpha^3 + \alpha + 1 = 0$, you found three roots of $T^3 + T + 1$ and three roots of $T^3 + T^2 + 1$ on Set 2 by brute force computations. Compute repeated squares of α to find all the roots of $T^3 + T + 1$ in a new way. Then do the same thing for $T^3 + T^2 + 1$ by starting with any one root of it in $\mathbf{F}_2[\alpha]$ and repeatedly squaring. This is a slicker method of finding all the roots!

4. (Ideals of continuous functions) Let $C([0, 1])$ denote the ring of continuous functions $[0, 1] \to \mathbf{R}$, with pointwise addition and multiplication: $(f + g)(t) = f(t) + g(t)$ and $(fg)(t) = f(t)g(t)$.

   a) For any nonempty subset $S \subset [0, 1]$, show the set of continuous functions $[0, 1] \to \mathbf{R}$ which vanish on that set is an ideal in $C([0, 1])$. It is denoted
   $$I(S) = \{f \in C([0, 1]) : f(x) = 0 \text{ for all } x \in S\}.$$
   Why don't we get an ideal using $\{f : f(x) = 1 \text{ for all } x \in S\}$?

   b) When $S = \{x_0\}$ is a single point, show $C([0, 1])/I(x_0) \cong \mathbf{R}$, so $I(x_0)$ is a maximal ideal. (Hint: Find a ring homomorphism $C([0, 1]) \to \mathbf{R}$ with image $\mathbf{R}$ and kernel $I(x_0)$.)

   c) For different points x_0 and x_1 in $[0, 1]$, explicitly construct continuous functions to show $I(x_0) \neq I(x_1)$. (Hint: You can find examples among linear functions. Find one that is 0 at one point and not 0 at the other.)

   Remark. The converse of part b is true: every maximal ideal in $C([0, 1])$ is some $I(x_0)$. So the points of $[0, 1]$ can be "reconstructed" from the ring $C([0, 1])$ by interpreting *points* as *maximal ideals*. This is a profound connection between analysis and abstract algebra.

5. Determine the degree and a basis for each of the following field extensions:
   $$\mathbf{Q}(\alpha)/\mathbf{Q} \text{ where } \alpha^3 - 9\alpha - 9 = 0, \quad \mathbf{Q}(\sqrt{2}, \sqrt[3]{5})/\mathbf{Q}(\sqrt{2}).$$