Math 3231 - Abstract Algebra II
Problem Set 1

Due in MSB 318
1/30/09 at 3 PM

Reading: §§4.1, 4.2, 4.3.

In most sciences one generation tears down what another has built and what one has established another undoes. In mathematics alone each generation builds a new story to the old structure. H. Hankel

1. Show the set
$$\mathbf{Q}[\sqrt{10}] = \{a + b\sqrt{10} : a, b \in \mathbf{Q}\},$$
which is easily closed under addition and subtraction, is also closed under multiplication and each nonzero element of $\mathbf{Q}[\sqrt{10}]$ has a multiplicative inverse in $\mathbf{Q}[\sqrt{10}]$, so it is a subfield of $\mathbf{R}$. Then show $\sqrt{2} \notin \mathbf{Q}[\sqrt{10}]$ by showing no element of $\mathbf{Q}[\sqrt{10}]$ squares to 2.

2. (A small table of irreducibles)

   a) Compute all the irreducible polynomials of degree up to 4 in $\mathbf{F}_2[T]$. Test a polynomial for irreducibility by checking it is not divisibile by lower-degree irreducibles. In your solution, present the list of irreducibles first, ordered by degree, and then show your work which explains how you found those polynomials (and eliminated the others as reducible).

   b) Use your table to construct an example of a field of size 16 and then find a nonzero element in this field which is a generator of the multiplicative group of the field.

3. Let's use division with remainder in $\mathbf{F}_3[T]$ to carry out procedures similar to what can be done with division with remainder in $\mathbf{Z}$.

   a) In $\mathbf{F}_3[T]$, solve each of the equations
$$T^3 + 2T + 2 = (T - 1)q(T) + r(T), \quad T^4 + 1 = (T^3 - T)Q(T) + R(T),$$
where the remainders $r(T)$ and $R(T)$ have degree less than 1 and 3, respectively.

   b) In $\mathbf{F}_3[T]$, compute the greatest common divisor of $T^5+2T^3+2T^2+2T$ and $T^4+2T^3+T^2+2$ using Euclid's algorithm.

4. In the field $\mathbf{F}_3[T]/(T^2 + 1)$, list the nonzero elements and determine the order of each one (multiplicatively).

5. For fields K and L, a *field homomorphism* $f : K \to L$ is a function satisfying $f(x + y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$ for all x and y in K and also $f(1) = 1$.

   a) Show the function $f : \mathbf{Q}[\sqrt{10}] \to \mathbf{Q}[\sqrt{10}]$ given by $f(a + b\sqrt{10}) = a - b\sqrt{10}$ is a field homomorphism.

   b) Show a field homomorphism is injective: if $f(x) = f(y)$ then $x = y$. (Hint: as in group theory, *explain* why it is enough to show if $f(x) = 0$ then $x = 0$. Then think about the fact that $f(1) = 1$ and any nonzero x is invertible, so $xy = 1$ for some y.)