Math 3231 - Abstract Algebra II
Midterm Review

To be discussed on Tuesday at 8 PM
in room to be determined

The main difficulty for the beginner is to absorb a reasonable vocabulary in a short time. None of the concepts is difficult, but there is an accumulation of new concepts which may sometimes seem heavy. S. Lang

1. (Ideals)

   a) Define ideals, principal ideals, and maximal ideals in a general commutative ring.

   b) If F is a field, E is a larger field, and $\alpha \in E$ is algebraic over F, define the minimal polynomial of α in $F[T]$.

   c) Show all ideals in $F[T]$ are principal and explain how this is related to the concept of minimal polynomial.

2. Let $f(T) \in F[T]$ be monic of degree n with a root α in some field E containing F.

   a) Letting $n = \deg f$, show $\{1, \alpha, \dots, \alpha^{n-1}\}$ is a spanning set for E as an F-vector space: every element of E has the form $a_0 + a_1\alpha + \cdots + a_{n-1}\alpha^{n-1}$ for some a_i's in F.

   b) Suppose $\alpha^4 + 2\alpha^2 - \alpha + 2 = 0$. Write α^6 as a linear combination of $\{1, \alpha, \alpha^2, \alpha^3\}$ by two methods: compute successively α^4, α^5, and α^6 in terms of lower powers using the equation $\alpha^4 + 2\alpha^2 - \alpha + 2 = 0$ and then by carrying out polynomial division of T^6 by $T^4 + 2T^2 - T + 2$ and setting $T = \alpha$ after you find the quotient and remainder.

   c) If $f(T)$ is irreducible in $F[T]$, use properties of polynomials in $F[T]$ to show $\{1, \alpha, \dots, \alpha^{n-1}\}$ is linearly independent over F. Make it very clear where you use irreducibility of $f(T)$.

3. Let $f(T) = T^3 + 3T - 1$ and $g(T) = T^3 - 3T - 1$. These polynomials superficially "look the same," but we will see in this problem (parts d and e) that their roots exhibit very different behavior.

   a) Show $f(T)$ and $g(T)$ are irreducible in $\mathbf{Q}[T]$ using, in both cases, the two standard tests: reduction mod p and Eisenstein translates.

   b) Sketch a graph of $y = f(x)$ and also a graph of $y = g(x)$. Then use calculus to justify what the graphs suggest: $f(T)$ has exactly one root in $\mathbf{R}$ and $g(T)$ has three roots in $\mathbf{R}$.

   c) Let α be a real root of $f(T)$ and let β be a real root of $g(T)$. Explicitly factor $f(T) = (T - \alpha)h(T)$ in $\mathbf{Q}(\alpha)[T]$ and $g(T) = (T - \beta)k(T)$ in $\mathbf{Q}(\beta)[T]$. (Find $h(T)$ and $k(T)$.)

   d) Show the only root of $f(T)$ in $\mathbf{Q}(\alpha)$ is α. (Hint: Any other root of $f(T)$ is a root of $h(T)$ from part c. Use your formula for $h(T)$ in part c to show $h(T)$ has no real root, and hence no root in $\mathbf{Q}(\alpha) \subset \mathbf{R}$.)

   e) Show $g(T)$ has more roots in $\mathbf{Q}(\beta)$ than just β by showing $2 - \beta^2$ is a root of $g(T)$ *and* that $2 - \beta^2 \neq \beta$ by linear algebra ideas (what is a $\mathbf{Q}$-basis for $\mathbf{Q}(\beta)/\mathbf{Q}$?).

   f) (Bonus) Find a third root of $g(T)$ in $\mathbf{Q}(\beta)$ besides β and $2 - \beta^2$.

4. Let $\alpha = \sqrt{2 + \sqrt{2}}$ and $\beta = \sqrt{2 - \sqrt{2}}$ in $\mathbf{R}$. All square roots appearing here are positive.

   a) Show α and β have the same minimal polynomial in $\mathbf{Q}[T]$.

   b) Let $f(T) \in \mathbf{Q}[T]$ be the common minimal polynomial in part a. Describe all the roots of $f(T)$ in terms of α and β, and then show α and β have different minimal polynomials in $\mathbf{Q}(\sqrt{2})[T]$.

   c) Show $\beta = \alpha^3 - 3\alpha$. Conclude that all roots of the minimal polynomial of α lie in the field $\mathbf{Q}(\alpha)$.

5. Let F be a field containing $\mathbf{F}_3$.

   a) Show the cubing function $\varphi(x) = x^3$ is a ring homomorphism $F \to F$.

   b) For $f(T) \in \mathbf{F}_3[T]$ and $x \in F$, show $f(x)^3 = f(x^3)$, using part a.

   c) Show the polynomial $T^3 - T - 1$ is irreducible in $\mathbf{F}_3[T]$, compute the size of the field $\mathbf{F}_3(\alpha)$, where α is a root of this cubic polynomial, and use part b to find all three roots of $T^3 - T - 1$ in $\mathbf{F}_3(\alpha)$ in terms of an $\mathbf{F}_3$-basis for $\mathbf{F}_3(\alpha)$. (Hint: You did something similar to this for field containing $\mathbf{F}_2$ on the last homework.)

6. (Examples) Give an example illustrating each of the following.

   a) A polynomial that is irreducible in $\mathbf{Q}[T]$ but is reducible mod 3.

   b) A domain that is not a field.

   c) A $\mathbf{Q}$-basis of the field $\mathbf{Q}(\sqrt[4]{2})$.

   d) A field of size 16.