The Dimension of a Vector Space
Math 316, UConn Spring 2004

This handout is a supplementary discussion leading up to the definition of dimension and some of its basic properties.

Let V be a vector space over a field F. For any subset $\{v_1, \dots, v_n\}$ of V, its *span* is the set of all of its linear combinations:

$$\mathrm{Span}(v_1, \dots, v_n) = \{c_1 v_1 + \cdots + c_n v_n : c_i \in F\}.$$

In $\mathbf{R}^3$, $\mathrm{Span}((1,0,0),(0,1,0))$ is the xy-plane in $\mathbf{R}^3$. If v is a single vector in V,

$$\mathrm{Span}(v) = \{cv : c \in F\} = Fv$$

is the set of scalar multiples of v.

Since sums of linear combinations are linear combinations and the scalar multiple of a linear combination is a linear combination, $\mathrm{Span}(v_1, \dots, v_n)$ is a subspace of V. It may not be the whole space, of course. If it is, that is, if *every* vector in V is a linear combination from $\{v_1, \dots, v_n\}$, we say this set *spans* V.

A subset $\{w_1, \dots, w_m\}$ of V is called *linearly independent* when the vanishing of a linear combination only happens in the obvious way:

$$c_1 w_1 + \cdots + c_m w_m = 0 \Longrightarrow \text{ all } c_i = 0.$$

From this condition, one sees that a linear combination of linearly independent vectors has only one possible set of coefficients:

$$c_1 w_1 + \cdots + c_m w_m = c_1' w_1 + \cdots + c_m' w_m \Longrightarrow \text{ all } c_i = c_i'. \tag{1}$$

Indeed, subtracting gives $\sum (c_i - c_i') w_i = 0$, so $c_i - c_i' = 0$ for all i by linear independence. Thus $c_i = c_i'$ for all i.

Spanning sets for V and linearly independent subsets of V are in some sense opposite concepts. Any subset of a linearly independent subset is still linearly independent, but this need not be true of spanning sets. Any superset of a spanning set for V is still a spanning set for V, but this need not be true of linearly independent subsets.

A subset of V which has both of the above properties is called a *basis*. That is, a basis of V is a linearly independent subset of V which also spans V. For most vector spaces, there is no God-given basis, so what matters conceptually is the size of a basis rather than a particular choice of basis (compare with the notion of a cyclic group with the particular choice of a generator).

The following theorem is a first result which links spanning sets in V with linearly independent subsets.

Theorem 1. *Suppose $V \neq \{0\}$ and it admits a finite spanning set $v_1, \dots, v_n$. Some subset of this spanning set is a linearly independent spanning set.*

The theorem says that once there is a finite spanning set, which could have lots of linear dependence relations, there is a basis for the space. Moreover, the theorem tells us a basis can be found within any spanning set at all.

Proof. While $\{v_1, \dots, v_n\}$ may not be linearly independent, it contains linearly independent subsets, such as any one single nonzero v_i. Of course, such small linearly independent subsets can hardly be expected to span V. But consider linearly independent subsets of $\{v_1, \dots, v_n\}$ which are *as large as possible*. Reindexing, without loss of generality, we can write such a subset as $\{v_1, \dots, v_k\}$.

For $i = k+1, \ldots, n$, the set $\{v_1, \ldots, v_k, v_i\}$ is not linearly independent (otherwise $\{v_1, \ldots, v_k\}$ is not a maximal linearly independent subset). Thus there is some linear relation

$$c_1v_1 + \cdots + c_kv_k + c_iv_i = 0,$$

where the c's are in F are not all of them are 0. The coefficient c_i cannot be zero, since otherwise we would be left with a linear dependence relation on $v_1, \ldots, v_k$, which does not happen due to their linear independence.

Since $c_i \neq 0$, we see that v_i is in the span of $v_1, \ldots, v_k$. This holds for $i = k+1, \ldots, n$, so any linear combination of $v_1, \ldots, v_n$ is also a linear combination of just $v_1, \ldots, v_k$. As every element of V is a linear combination of $v_1, \ldots, v_n$, we conclude that $v_1, \ldots, v_k$ spans V. By its construction, this is a linearly independent subset of V as well. □

Notice the *non-constructive* character of the proof. If we somehow can check that a (finite) subset of V spans the whole space, Theorem 1 says a subset of this is a linearly independent spanning set, but the proof is not constructively telling us which subset of $\{v_1, \ldots, v_n\}$ this might be.

Theorem 1 is a "top-down" theorem. It says any (finite) spanning set has a linearly independent spanning set inside of it. It is natural to ask if we can go "bottom-up," and show any linearly independent subset can be enlarged to a linearly independent spanning set. Something along these lines will be proved in Corollary 3.

Lemma 1. *Suppose $\{v_1, \ldots, v_n\}$ spans V, where $n \geq 2$. Pick any $v \in V$. If some v_i is a linear combination of the other v_j's with v, then V is spanned by those other vectors with v.*

For example, if V is spanned by v_1, v_2, and v_3, and v_1 is a linear combination of v, v_2, and v_3, where v is another vector in V, then V is spanned by v, v_2, and v_3.

Lemma 1 should be geometrically reasonable. See if you can prove it before reading the proof below.

Proof. Reindexing if necessary, we can suppose it is v_1 which is a linear combination of $v, v_2, \ldots, v_n$. We will show every vector in V is a linear combination of $v, v_2, \ldots, v_n$, so these vectors span V.

Pick any $w \in V$. By hypothesis,

$$w = c_1v_1 + c_2v_2 + \cdots + c_nv_n$$

for some $c_i \in F$. Since v_1 is a linear combination of $v, v_2, \ldots, v_n$, we feed this linear combination into the above equation to see w is a linear combination of $v, v_2, \ldots, v_n$. As w was arbitrary in V, we have shown V is spanned by $v, v_2, \ldots, v_m$. □

The following important technical result relates spanning sets for V and linearly independent subsets of V. It is called the exchange theorem. The name arises from the process in its proof, which relies on repeated applications of Lemma 1.

Theorem 2 (Exchange Theorem)**.** *Suppose V is spanned by n vectors, where $n \geq 1$. Any linearly independent subset of V has at most n vectors.*

If you think about linear independence as "degrees of freedom," the exchange theorem makes sense. What makes the theorem somewhat subtle to prove is that the theorem bounds the size of any linearly independent subset once we know the size of one spanning set. Most linearly independent subsets of V are not directly related to the given choice of spanning set, so linking the two sets of vectors is tricky. The proof will show how to link linearly independent sets and spanning sets by an exchange process, one vector at a time.

Proof. First, let's check the result when $n = 1$. In this case, $V = Fv$ for some v (that is, V is spanned by one vector). Two different scalar multiples of v are linearly dependent, so a linearly independent subset of V can have size at most 1.

Now we take $n \geq 2$. We give a proof by *contradiction*. If the theorem is false, then V contains a set of $n+1$ linearly independent vectors, say $w_1, \ldots, w_{n+1}$.

Step 1: We are told that V can be spanned by n vectors. Let's call such a spanning set $v_1, \ldots, v_n$. We also have the $n+1$ linearly independent vectors $w_1, \ldots, w_{n+1}$ in V. Write the first vector from our linearly independent set in terms of our spanning set:

$$w_1 = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$$

for some $c_i \in F$. Since $w_1 \neq 0$ (a linearly independent set never contains the vector 0), some coefficient c_j is nonzero. Without loss of generality, we can reindex the v's so that c_1 is nonzero. Then the above equation can be solved for v_1 as a linear combination of $w_1, v_2, \ldots, v_n$. By Lemma 1,

$$V = \operatorname{Span}(w_1, v_2, \ldots, v_n). \tag{2}$$

Notice that we have taken one element from the initial spanning set out and inserted an element from the linearly independent set in its place, retaining the spanning property.

Step 2: Let's repeat the procedure, this time using our new spanning set (2). Write w_2 in terms of this new spanning set:

$$w_2 = c_1' w_1 + c_2' v_2 + \cdots + c_n' v_n \tag{3}$$

for some c_i' in F. We want to use this equation to show w_2 can be inserted into (2) and one of the original vectors can be taken out, without destroying the spanning property. Some care is needed, because we want to keep w_1 in the spanning set rather than accidentally swap it out. (This is an issue that we did not meet in the first step, where no new vectors had yet been placed in the spanning set.)

Certainly one of $c_1', c_2', \ldots, c_n'$ is nonzero, since w_2 is nonzero. But in fact we can say something a bit sharper: regardless of the value of c_1', one of $c_2', \ldots, c_n'$ is nonzero. Indeed, if $c_2', \ldots, c_m'$ are all zero, then $w_2 = c_1' w_1$ is a scalar multiple of w_1, and that violates linear independence (as $\{w_1, \ldots, w_m\}$ is linearly independent, so is the subset $\{w_1, w_2\}$).

Without loss of generality, we can reindex $v_2, \ldots, v_n$ so it is c_2' which is nonzero. Then we can use (3) to express v_2 as a linear combination of $w_1, w_2, v_3, \ldots, v_n$. By another application of Lemma 1, using our new spanning set in (2) and the auxiliary vector w_2, it follows that

$$V = \operatorname{Span}(w_1, w_2, v_3, \ldots, v_n).$$

Step 3: Now that we see how things work, we argue inductively.

Suppose for some k between 1 and $n-1$ that we have shown

$$V = \operatorname{Span}(w_1, \ldots, w_k, v_{k+1}, \ldots, v_n).$$

(This has already been checked for $k = 1$ in Step 1, and $k = 2$ in Step 2, although Step 2 is not logically necessary for what we do; it was just included to see concretely the inductive step we now carry out for any k.)

Using this spanning set for V, write

$$w_{k+1} = a_1 w_1 + \cdots + a_k w_k + a_{k+1} v_{k+1} + \cdots + a_n v_n \tag{4}$$

with $a_i \in F$. One of $a_{k+1}, \ldots, a_n$ is nonzero, since otherwise this equation expresses w_{k+1} as a linear combination of $w_1, \ldots, w_k$, and that violates linear independence of the w's.

Reindexing $v_{k+1}, \ldots, v_n$ if necessary, we can suppose it is a_{k+1} which is nonzero. Then (4) can be solved for v_{k+1} as a linear combination of $w_1, \ldots, w_k, w_{k+1}, v_{k+2}, \ldots, v_n$. By Lemma 1, using the spanning set $\{w_1, \ldots, w_k, v_{k+1}, \ldots, v_n\}$ and the auxiliary vector w_{k+1}, we can swap w_{k+1} into the spanning set in exchange for v_{k+1} without losing the spanning property:

$$V = \mathrm{Span}(w_1, \dots, w_k, w_{k+1}, v_{k+2}, \dots, v_n).$$

We have added an new vector to the spanning set and taken one of the original vectors out. Now by induction (or, more loosely, "repeating this step $n-k-1$ more times"), we arrive at the conclusion that

$$V = \mathrm{Span}(w_1, \dots, w_n). \tag{5}$$

However, we were starting with $n+1$ linearly independent vectors $w_1, \dots, w_{n+1}$, so w_{n+1} is not in the span of $w_1, \dots, w_n$. That contradicts the meaning of (5). We have reached a contradiction, so no linearly independent subset of V contains more than n vectors, where n is the size of some spanning set for V. □

As an example, since the vector space $\mathrm{M}_3(\mathbf{R})$ of 3×3 real matrices has a 9 element spanning set (the 9 matrices with 1 in one component and 0 elsewhere), any linearly independent subset of $\mathrm{M}_3(\mathbf{R})$ has at most 9 elements in it.

Corollary 1. *Suppose V admits a finite basis. Any two bases for V have the same size.*

Proof. Let $\{v_1, \dots, v_n\}$ and $\{v'_1, \dots, v'_m\}$ be bases for V. Treating the first set as a spanning set for V and the second set as a linearly independent subset of V, the exchange theorem tells us that $m \leq n$. Reversing these roles (which we can do since bases are both linearly independent and span the whole space), we get $n \leq m$. Thus $m = n$. □

The (common) size of any basis of V is called the *dimension* of V (over F). For example, from the known bases of $\mathbf{R}^n$ and $\mathrm{M}_n(\mathbf{R})$, these spaces have real dimension n and n^2, respectively. A basis of $\mathbf{C}$ as a real vector space is 2, with one basis being $\{1, i\}$.

Theorem 3. *Let V be a vector space with dimension n. Any spanning set has at least n elements, and contains a basis inside of it. Any linearly independent subset has at most n elements, and can be extended to a basis of V. Finally, an n-element subset of V is a spanning set if and only if it is a linearly independent set.*

Proof. Since V has a basis of n vectors, let's pick such a basis, say $v_1, \dots, v_n$. We will compare this basis to the spanning sets and the linearly independent sets to draw our conclusions, taking advantage of the dual nature of a basis as *both* a linearly independent subset of V and as a spanning set for V.

If $\{u_1, \dots, u_k\}$ is a spanning set for V, then a comparison with $\{v_1, \dots, v_n\}$ (interpreted as a linearly independent subset of V) shows $n \leq k$ by the exchange theorem. Equivalently, $k \geq n$. Moreover, Theorem 1 says That $\{u_1, \dots, u_k\}$ contains a basis for V. This settles the first part of the theorem.

For the next part, suppose $\{w_1, \dots, w_m\}$ is a linearly independent subset of V. A comparison with $\{v_1, \dots, v_n\}$ (interpreted as a spanning set for V) shows $m \leq n$ by the exchange theorem. To see that the w's can be extended to a basis of V, apply the exchange process from the proof of the exchange theorem, but only m times since we have only m linearly independent w's. We find at the end that

$$V = \mathrm{Span}(w_1, \dots, w_m, v_{m+1}, \dots, v_n),$$

which shows the w's can be extended to a spanning set for V. This spanning set contains a basis for V, by Theorem 1. Since all bases of V have n elements, this n-element spanning set must be a basis, so it is a linearly independent set.

Taking $m = n$ in the previous paragraph shows any n-element linearly independent subset is a basis (and thus spans V). Conversely, any n-element spanning set is linearly independent, since any linear dependence relation would let us cut down to a spanning set of fewer than n elements, but that violates what we showed earlier in this proof. □

Theorem 4. *If V is an n-dimensional vector space, any subspace W is finite-dimensional, with dimension at most n.*

Proof. Any linearly independent subset of W is also a linearly independent subset of V, and thus has size at most n by Theorem 3. Choose a linearly independent subset $\{w_1, \ldots, w_m\}$ of W with a *maximal* value for $m \leq n$. Then the w_i's span W. Indeed, augmenting this subset by any other vector $w \in W$ must give a linearly dependent set, or otherwise the maximality condition on $\{w_1, \ldots, w_m\}$ is violated. Writing out a linear dependence relation on the w_i's and w, the coefficient of w has to be nonzero, since the w_i's are a linearly independent set on their own. Divide through by that nonzero coefficient to see any $w \in W$ is a linear combination of $w_1, \ldots, w_m$. Therefore these m vectors are a basis of W: they are linearly independent by construction, and we proved that they span W. Thus W is finite-dimensional, with dimension $m \leq n$. □

Theorem 5. *If V has dimension n and W is a subspace with dimension n, then $W = V$.*

Proof. When W has dimension n, any basis for W is a linearly independent subset of V with n elements, so it spans V by Theorem 3. The span is also W (by definition of a basis for W), so $W = V$. □