GENERALIZED ADDITION

YOUR NAME

Math 2784 (or 2794W)

Date: Date Here.
1. Introduction

In special relativity, velocities are constrained to an interval \((-c, c)\) where \(c\) is the speed of light\(^1\) and velocities are combined by the formula

\[
v \oplus w = \frac{v + w}{1 + vw/c^2}.
\]

This operation was introduced by Einstein (Figure 1). In Table 1 we see how velocities add when they are expressed as fractions of \(c\). The results remain less than \(c\).

\[
\begin{array}{ccc}
v & w & v \oplus w \\
(3/4)c & (1/2)c & (10/11)c \\
(3/4)c & (9/10)c & .98507c \\
(9/10)c & (99/100)c & .99947c \\
\end{array}
\]

Table 1. Adding velocities in \((-c, c)\) relativistically.

While \(\oplus\) on \((-c, c)\) may seem unusual, it has some similar properties to addition on \(\mathbb{R}\):

- Closure, \(i.e.,\) if \(v_1, v_2 \in (-c, c)\) then \(v_1 \oplus v_2 \in (-c, c)\).
- Identity for \(\oplus\): \(0 \oplus v = v \oplus 0 = v\) for \(v \in (-c, c)\).
- Inverse of any \(v\) under \(\oplus\) is \(-v\): \(v \oplus -v = -v \oplus v = 0\).
- Associativity: \((v_1 \oplus v_2) \oplus v_3 = v_1 \oplus (v_2 \oplus v_3)\) for any \(v_1, v_2, v_3 \in (-c, c)\).

If we define \(L: (-c, c) \rightarrow \mathbb{R}\) by

\[
L(v) = \frac{1}{2} \ln \left( \frac{1 + v/c}{1 - v/c} \right),
\]

then this “rescaled” velocity turns \(\oplus\) into ordinary addition:

\[
L(v \oplus w) = L(v) + L(w).
\]

This means that, in a sense, \(L\) is a “relativistic logarithm.” Its inverse, a “relativistic exponential,” is essentially a hyperbolic tangent function [3].

It turns out that any reasonable method of combining numbers on an interval, not just the operation on \((-c, c)\) in (1.1), has an analogue of the logarithm: a rescaling function that turns the operation into ordinary addition on \(\mathbb{R}\).

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\(^1\)Mathematically, for our purposes \(c\) can be any positive number.
2. Generalized addition on an interval

Let $I \subset \mathbb{R}$ be an open interval. Assume we have a binary operation $*: I \times I \to I$ with the following four properties:

1. **Closure.** $x, y \in I \implies x*y \in I$.
2. **Identity.** There is $u \in I$ such that for any $x \in I$, $x*u = u*x = x$.
3. **Inverses.** For any $x \in I$ there is some $i(x) \in I$ such that $x*i(x) = i(x)*x = u$.
4. **Associativity.** For $x, y, z \in I$, $(x*y)*z = x*(y*z)$.

To state our main theorem, it is useful to write the operation $x*y$ using the notation of a function of two variables: set $F(x, y) = x*y$. Two examples of this are

$$F(v, w) = \begin{cases} v + w, & \text{if } I = \mathbb{R}, \\ \frac{v+w}{1+w/c^2}, & \text{if } I = (-c, c). \end{cases}$$

The above properties of $*$ take the following form using the functional notation $F$:

1. **Closure.** $x, y \in I \implies F(x, y) \in I$.
2. **Identity.** There is $u \in I$ such that for any $x \in I$, $F(x, u) = F(u, x) = x$.
3. **Inverses.** For any $x \in I$ there is some $i(x) \in I$ such that $F(x, i(x)) = F(i(x), x) = u$.
4. **Associativity.** For $x, y, z \in I$, $F(F(x, y), z) = F(x, F(y, z))$.

**Theorem 2.1.** Suppose $F(x, y) = x*y$ is as above and has continuous partial derivatives. Write $F_1(x, y)$ for $\partial F/\partial x$. Then the function

$$(2.1) \qquad \ell(x) \overset{\text{def}}{=} \int_u^x dt \quad F_1(u, t)$$

is a differentiable bijection from $I$ to $\mathbb{R}$ that converts $*$ on $I$ to ordinary addition on $\mathbb{R}$. That is, $\ell(x*y) = \ell(x) + \ell(y)$.

The rescaling function $\ell$ is a ‘logarithm’ for $*$, since it turns $*$ on $I$ into addition on $\mathbb{R}$.

**Proof.** We briefly indicate the main steps. See [1] Chap. 6] or [2] for more details and for the motivation for formula (2.1).

The first step is to show $F_1(u, t) > 0$ for all $t \in I$, so the integral defining $\ell(x)$ makes sense. This uses the associativity and continuity of $F(x, y)$. In order to show $\ell(x*y) = \ell(x) + \ell(y)$, or equivalently $\ell(F(x, y)) = \ell(x) + \ell(y)$, we show the $x$-derivative of both sides is the same for all $x$, which relies on the Fundamental Theorem of Calculus.

Since $\ell(x) + \ell(i(x)) = \ell(x*i(x)) = \ell(u) = 0$, if $\ell(x) \neq 0$ then $\ell(x)$ and $\ell(i(x))$ have opposite sign. For any positive integer $n$,

$$\ell(x*\cdots*x) = n\ell(x), \quad \ell(i(x)*\cdots*i(x)) = n\ell(i(x)).$$

As $n \to \infty$, one of these tends to $\infty$ and the other to $-\infty$. Since $\ell(I)$ is an interval by continuity of $\ell$ and the Intermediate Value Theorem, we must have $\ell(I) = \mathbb{R}$.

**References**

