

RELATIVISTIC ADDITION AND REAL ADDITION

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1. INTRODUCTION

For three particles P, Q, R travelling on a straight line, let v_{PQ} be the (relative) velocity of P as measured by Q , and define v_{QR}, v_{PR} similarly.

According to classical mechanics, the velocity v of a particle moving on a line can be any real number, and relative velocities add by the simple formula

$$(1.1) \quad v_{PR} = v_{PQ} + v_{QR}.$$

On the other hand, the special theory of relativity says velocities are restricted to a bounded range, $-c < v < c$, where c is the speed of light (whose value of course depends on the choice of units, and it is convenient to choose them so $c = 1$, but we won't do that.) The relativistic addition formula for velocities is:

$$(1.2) \quad v_{PR} = \frac{v_{PQ} + v_{QR}}{1 + (v_{PQ}v_{QR}/c^2)}.$$

Example 1.1. If $v_{PQ} = \frac{3}{4}c$ and $v_{QR} = \frac{1}{2}c$ then a comparison of the right sides of (1.1) and (1.2) gives

$$v_{PQ} + v_{QR} = \frac{5}{4}c > c, \quad \frac{v_{PQ} + v_{QR}}{1 + (v_{PQ}v_{QR}/c^2)} = \frac{10}{11}c < c.$$

While the first formula gives a value which exceeds c , the second formula gives a value which is less than c .

Although (1.1) and (1.2) appear different, when both velocities are *small* they are approximately the same. Indeed, if $v_{PQ}, v_{QR} \ll c$ then $v_{PQ}v_{QR}/c^2 \approx 0$ and the denominator in (1.2) is nearly 1, so (1.2) is approximately equal to its numerator, which is the right side of (1.1).

If only one velocity is small compared to c , then the difference between (1.1) and (1.2) can still be noticed if we think about the relative error compared to the size of the smaller velocity. Table 1 gives in the last column the difference between classical and relativistic formulas for v_{PR} . Note there is significant *relative error* not only in the first row, when both v_{PQ} and v_{QR} are substantial fractions of the speed of light, but even in the second and third rows, when only v_{PQ} is a substantial fraction of c . Specifically, for small v_{QR}/c the fourth column entry is about $.5625v_{QR} = (9/16)v_{QR}$, which is a not insubstantial fraction of v_{QR} .

v_{PQ}	v_{QR}	(1.2)	(1.1) - (1.2)
$(3/4)c$	$(1/2)c$	$(10/11)c$	$.341c$
$(3/4)c$	$(1/100)c$	$.75434c$	$.00566c$
$(3/4)c$	$(1/1000)c$	$.750437c$	$.000563c$

TABLE 1. Classical and Relativistic Comparisons

We can use a Taylor expansion to understand why (1.2) noticeably deviates from (1.1) as soon as just one of the velocities is not small compared to c . For $v, w \in (-c, c)$, fix v and think of $(v + w)/(1 + vw/c^2)$ as a function of w . Call it $h(w)$. For w/c small, a Taylor expansion at 0 yields

$$\frac{v + w}{1 + vw/c^2} = h(w) \approx h(0) + h'(0)w = v + \left(1 - \frac{v^2}{c^2}\right)w.$$

If v/c is also small, the coefficient of w is about 1, so $(v + w)/(1 + vw/c^2) \approx v + w$. But if v/c is not small, *e.g.*, $v = \frac{3}{4}c$, then we see a deviation from the classical velocity addition

rule $v + w$ by an error of around $(v/c)^2 w$. This explains the error $\frac{9}{16}v_{QR}$ in the table above, where $v = \frac{3}{4}c$.

Our goal in this paper is not to focus on the differences between (1.1) and (1.2), but on their similarities through common algebraic properties. The classical model for velocity allows all real numbers as velocities and combines them under addition. Special relativity involves velocities in an interval $(-c, c)$ and combines them by the formula

$$(1.3) \quad v \oplus w := \frac{v + w}{1 + vw/c^2}.$$

The following algebraic properties of the operation \oplus are similar to properties of ordinary addition on \mathbf{R} :

- Closure: $v, w \in (-c, c) \Rightarrow v \oplus w \in (-c, c)$.
- Identity for \oplus : $0 \oplus v = v \oplus 0 = v$ for $v \in (-c, c)$.
- Inverse of any v under \oplus is $-v$: $v \oplus -v = -v \oplus v = 0$.
- Associativity: $(v \oplus w) \oplus x = v \oplus (w \oplus x)$ for $v, w, x \in (-c, c)$.

It is left to the reader to check these, of which the first and fourth are the only ones with much content. Note usual addition is *not* closed on $(-c, c)$, as we saw in Example 1.1.

In addition to the operation \oplus on $(-c, c)$, the operation of multiplication on the interval $(0, \infty)$ of positive real numbers also has the four properties above (where the identity is 1 and the inverse of x is $1/x$), and moreover we know the *logarithm* turns multiplication on $(0, \infty)$ into addition on \mathbf{R} : $\log(xy) = \log x + \log y$, where \log means the natural logarithm (or use whatever logarithmic base you want). It turns out that there is an analogous “relativistic logarithm” $L(v)$ defined on numbers $v \in (-c, c)$ which turns \oplus on $(-c, c)$ into addition on \mathbf{R} :

$$L(v \oplus w) = L(v) + L(w).$$

In other words, relativistic addition on $(-c, c)$ is just a disguised form of ordinary addition on \mathbf{R} , and the relativistic logarithm (which we have not yet written down as an explicit formula, but will do this later on) unmasks this relationship.

In the remainder of this paper we will figure out a formula for the relativistic logarithm in Section 2 and then show in Section 3, quite generally, that any reasonable “addition-like” operation on any interval of real numbers has its own special log-like function which reveals the operation to be ordinary addition in a hidden form.

2. RESCALING WITH THE HYPERBOLIC TANGENT

The formula for $v \oplus w$ in (1.3) is reminiscent of the addition formula for the tangent function:

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - (\tan x)(\tan y)}.$$

However, there is a minus sign in the denominator here where there is a plus sign in the denominator of (1.3).

The *hyperbolic* tangent is better. It is defined by the formula

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}},$$

where x is any real number. (A discussion of the hyperbolic functions is in Appendix A. In particular, their strong similarities to the classical trigonometric functions are seen there

using calculus.) The values of $\tanh x$ lie in the interval $(-1, 1)$ and a little algebra shows

$$\tanh(x + y) = \frac{\tanh x + \tanh y}{1 + (\tanh x)(\tanh y)}.$$

This is *exactly* like (1.3), up to factors of c . Let's scale the hyperbolic tangent by c to make its values lie in $(-c, c)$ instead of in $(-1, 1)$. The function $\varphi(x) := c \tanh(x)$ has values in $(-c, c)$ and

$$(2.1) \quad \varphi(x + y) = \varphi(x) \oplus \varphi(y).$$

This should be considered an analogue for \oplus on $(-c, c)$ of the exponential rule

$$e^{x+y} = e^x e^y.$$

What we actually care more about is not a way of transforming $+$ into \oplus but \oplus into $+$ (turn the more complicated operation into an easier operation). For this, we should use the inverse of the hyperbolic tangent. To find it, we carry out some algebra: if $y = \tanh x$ (so $-1 < y < 1$), we want to write x in terms of y . Since

$$\begin{aligned} y &= \frac{e^x - e^{-x}}{e^x + e^{-x}} \\ &= \frac{e^{2x} - 1}{e^{2x} + 1}, \end{aligned}$$

clearing denominators gives

$$ye^{2x} + y = e^{2x} - 1.$$

Collecting the x -terms on the left,

$$e^{2x}(y - 1) = -1 - y.$$

Dividing and taking care of sign issues,

$$e^{2x} = \frac{1 + y}{1 - y}.$$

Taking natural logarithms of both sides and dividing by 2,

$$(2.2) \quad x = \frac{1}{2} \log \left(\frac{1 + y}{1 - y} \right).$$

Since $y = \tanh x$, (2.2) gives us the inverse hyperbolic tangent function:

$$\tanh^{-1}(y) = \frac{1}{2} \log \left(\frac{1 + y}{1 - y} \right)$$

when $-1 < y < 1$.

The “ c -scaled” version of the inverse hyperbolic tangent, which is defined on $(-c, c)$ instead of on $(-1, 1)$, is

$$L(v) = \frac{1}{2} \log \left(\frac{1 + v/c}{1 - v/c} \right).$$

Since $L(\varphi(x)) = x$, where $\varphi(x) = c \tanh x$, applying L to both sides of (2.1) shows L turns \oplus into ordinary addition, just as the usual logarithm turns multiplication of positive numbers into ordinary addition:

$$L(v \oplus w) = L(v) + L(w).$$

Thus by the clever transformation $L(v)$, which is essentially the inverse of the hyperbolic tangent of v , we can replace velocities $v \in (-c, c)$ by rescaled velocities $L(v) \in \mathbf{R}$ and this converts \oplus on $(-c, c)$ into ordinary addition of real numbers.

Example 2.1. Using the values from Example 1.1, if $v = \frac{3}{4}c$ and $w = \frac{1}{2}c$ then $L(v) = \frac{1}{2} \log 7$, $L(w) = \frac{1}{2} \log 3$, and $L(v \oplus w) = L(\frac{10}{11}c) = \frac{1}{2} \log 21 = L(v) + L(w)$.

3. MORE GENERAL ADDITION LAWS

Rather than just looking at \oplus on $(-c, c)$, let's consider any open interval $I \subset \mathbf{R}$ with an “addition law” written as $*$. That is, $*$ has the following four properties:

- Closure: $x, y \in I \implies x * y \in I$.
- Identity: there is $u \in I$ such that for any $x \in I$, $x * u = u * x = x$.
- Inverses: for any $x \in I$ there is some $i(x) \in I$ such that $x * i(x) = i(x) * x = u$.
- Associativity: for $x, y, z \in I$, $(x * y) * z = x * (y * z)$.

Example 3.1. Take $I = (0, \infty)$ and $x * y = xy$, so “addition” is really ordinary multiplication. It has the four properties above, with $i(x) = 1/x$.

It will be useful to write the operation $x * y$ in the notation of a function of two variables: $F(x, y) = x * y$. For example, the classical and relativistic velocity addition formulas are

$$F(v, w) = v + w, \quad I = \mathbf{R}; \quad F(v, w) = \frac{v + w}{1 + vw/c^2}, \quad I = (-c, c).$$

The above properties of $*$ take the following form in terms of F :

- $x, y \in I \implies F(x, y) \in I$.
- There is $u \in I$ such that for any $x \in I$, $F(x, u) = F(u, x) = x$.
- For any $x \in I$ there is some $i(x) \in I$ such that $F(x, i(x)) = F(i(x), x) = u$.
- For $x, y, z \in I$, $F(F(x, y), z) = F(x, F(y, z))$.

Our main goal is to prove the following theorem, which says that under a suitable differentiability assumption this “addition law” on I admits a “logarithm” that turns it into ordinary addition on \mathbf{R} .

Theorem 3.2. *If $F(x, y) = x * y$ has continuous partial derivatives in x and y then there is a differentiable function $f: I \rightarrow \mathbf{R}$ that converts $*$ on I to ordinary addition on \mathbf{R} . That is, f is a differentiable bijection and $f(x * y) = f(x) + f(y)$.*

Theorem 3.2 will be proved by giving an *explicit recipe* for f . But we don't just want to write down a formula and blindly check it works. We want to understand where it comes from. Our treatment follows [3, pp. 164–167].

To discover f , rewrite the desired equation $f(x * y) = f(x) + f(y)$ as

$$f(F(x, y)) = f(x) + f(y).$$

Let's differentiate this equation with respect to x :

$$f'(F(x, y))F_1(x, y) = f'(x),$$

where we write $F_1(x, y)$ for the derivative with respect to the first variable x . Similarly we will write $F_2(x, y)$ for the derivative with respect to the second variable y . Setting $x = u$ (the identity for $*$), $F(x, y) = F(u, y) = u * y = y$, so

$$f'(y)F_1(u, y) = f'(u).$$

Let's now solve for $f'(y)$ and integrate:

$$(3.1) \quad f(y) = \int_u^y \frac{f'(u)}{F_1(u, t)} dt.$$

This is a formula for the rescaling function f . The constant $f'(u)$ is just an arbitrary factor, which we will choose to be 1 when we finally make a definition of $f(x)$. When we integrated in (3.1), we didn't introduce an additive constant since we want $f(u) = 0$ (just like $\log 1 = 0$) and the integral formula in (3.1) has this feature already. Of more pressing interest is the validity of dividing by $F_1(u, t)$ in (3.1). Why is it never zero?

Lemma 3.3. *For any $t \in I$, $F_1(u, t) > 0$.*

Proof. We differentiate the associative law, $x * (y * z) = (x * y) * z$, with respect to x . Using the F -notation instead of the $*$ -notation, recall associativity of $*$ says $F(F(x, y), z) = F(x, F(y, z))$. Differentiating both sides of this with respect to x ,

$$F_1(F(x, y), z)F_1(x, y) = F_1(x, F(y, z)).$$

Setting $x = u$, $F(x, y)$ becomes y :

$$(3.2) \quad F_1(y, z)F_1(u, y) = F_1(u, F(y, z)) = F_1(u, y * z).$$

So if $F_1(u, y) = 0$ for some y , then $F_1(u, y * z) = 0$ for any z . Choose $z = i(y)$ to get $F_1(u, u) = 0$. But this is not true:

$$(3.3) \quad F(x, u) = x \text{ for all } x \Rightarrow F_1(x, u) = 1 \text{ for all } x \Rightarrow F_1(u, u) = 1.$$

So $F_1(u, y)$ is nonzero for every y . Since it equals 1 at $y = u$, it must always be positive by the Intermediate Value Theorem (F_1 is continuous). \square

This lemma allows us to divide by $F_1(u, t)$ for any $t \in I$, and we will do this often without explicitly appealing to the lemma each time.

Since $1/F_1(u, t)$ is continuous in t , hence integrable, we are justified in making the following *definition*, for any x in the interval I :

$$(3.4) \quad f(x) \stackrel{\text{def}}{=} \int_u^x \frac{dt}{F_1(u, t)}.$$

By the fundamental theorem of calculus, f is differentiable and

$$(3.5) \quad f'(x) = \frac{1}{F_1(u, x)}.$$

In particular, $f'(u) = 1$ by (3.3).

With the choice of (3.4) for our rescaling function we now prove our main result Theorem 3.2.

Proof. We need to check two things:

- $f(F(x, y)) = f(x) + f(y)$.
- $f: I \rightarrow \mathbf{R}$ is a bijection.

For the first item, fix $y \in I$. We consider the x -derivatives of the two functions

$$f(F(x, y)), \quad f(x) + f(y).$$

By (3.5), the derivative of the first function is

$$f'(F(x, y))F_1(x, y) = \frac{F_1(x, y)}{F_1(u, F(x, y))}.$$

Does this equal the x -derivative of the second function, namely $f'(x) = 1/F_1(u, x)$? Setting them equal, we want to consider:

$$F_1(x, y)F_1(u, x) \stackrel{?}{=} F_1(u, F(x, y)).$$

This is just (3.2) with x, y, z relabelled as u, x, y . Therefore $f(F(x, y))$ and $f(x) + f(y)$ have equal x -derivatives for all x . Since they are equal at $x = u$, they are equal for all x . This verifies the first item.

For the second item, bijectivity of $f: I \rightarrow \mathbf{R}$, since $f'(y) = 1/F_1(u, y) > 0$ we see f is increasing, hence injective. To show surjectivity, note $f(I)$ is an interval by continuity. Choose $x \in I$, $x \neq u$. Since $f(x) + f(i(x)) = f(x * i(x)) = f(u) = 0$, $f(x)$ and $f(i(x))$ have opposite sign, one positive and the other negative. For any positive integer n ,

$$f(\underbrace{x * \cdots * x}_{n \text{ times}}) = nf(x), \quad f(\underbrace{i(x) * \cdots * i(x)}_{n \text{ times}}) = nf(i(x)),$$

As $n \rightarrow \infty$, one of these tends to ∞ and the other to $-\infty$. Since $f(I)$ is an interval, we must have $f(I) = \mathbf{R}$. \square

Corollary 3.4. *When the operation $x * y = F(x, y)$ has continuous partial derivatives, it is commutative. In particular, $F_1(x, y) = F_2(y, x)$ for all $x, y \in I$.*

Proof. Commutativity was never used in the proof of Theorem 3.2, so commutativity of ordinary addition on \mathbf{R} implies commutativity of $*$ on I . That is, in terms of the rescaling function (“logarithm”) f from I to \mathbf{R} , for any x and y in I we have

$$\begin{aligned} f(x * y) &= f(x) + f(y) \\ &= f(y) + f(x) \\ &= f(y * x), \end{aligned}$$

so $x * y = y * x$ since f is injective.

Now differentiate both sides of the formula $x * y = y * x$ with respect to x , rewriting this commutativity formula first as $F(x, y) = F(y, x)$ to make the notation easier to use in the context of calculus. We obtain $F_1(x, y) = F_2(y, x)$. \square

Corollary 3.4 says there is no way to define a non-commutative “addition law” $*$ on an interval of \mathbf{R} , granting reasonable differentiability assumptions on $x * y$ as a function of x and y . (The same conclusion holds if we grant only assumptions of continuity for $x * y$ as a function of x and y , but the proof then becomes more difficult. See [2].) For comparison, once we move beyond intervals and allow two degrees of freedom, non-commutative “addition laws” are possible. For instance, multiplication of the matrices $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ where $a > 0$ and $b \in \mathbf{R}$ is non-commutative but still shares the four basic properties from the start of this section.

Corollary 3.5. *The rescaling function f determines the operation $*$ by*

$$x * y = f^{-1}(f(x) + f(y)).$$

Proof. Apply f^{-1} to both sides of $f(x * y) = f(x) + f(y)$. \square

Since f is determined by the function $F_1(u, t)$ (Theorem 3.2), and the operation $x * y$ is determined by f (Corollary 3.5), we see the operation $*$ is encoded in the function $F_1(u, t) = F_2(u, t)$.

Choosing coordinates so $u = 0$, the function $F_1(u, x) = F_1(0, x)$ appears in the first term of the Taylor expansion at $y = u$ of $x * y = F(x, y)$ for small y :

$$(3.6) \quad x * y = F(x, y) \approx F(x, 0) + F_2(x, 0)y = x + F_1(0, x)y.$$

Since $F_1(0, 0) = 1$, for small x and y we get from (3.6) that $x * y \approx x + y$. But for small y and not-so-small x there is a deviation of $x * y$ from the simple law $x + y$, measured by the function $F_1(0, x)$. This deviation of $x * y$ from $x + y$ for any x and small y , which we already noticed in the example of relativistic addition in Table 1, can be used through its appearance in the function $f(x)$ to reconstruct the operation $x * y$ for any x and any y .

Let's look at some examples, to see which functions rescale various "addition laws" on an interval to ordinary addition on the real line.

Example 3.6. If $F(x, y) = x + y$ on $I = \mathbf{R}$, then $u = 0$, $F_1(0, x) = 1$, and

$$f(x) = \int_0^x \frac{dt}{F_1(0, t)} = \int_0^x dt = x.$$

Example 3.7. If $F(v, w) = v \oplus w = \frac{v + w}{1 + vw/c^2}$ on $(-c, c)$, then $u = 0$, $F_1(0, v) = 1 - \frac{v^2}{c^2}$, and

$$f(v) = \int_0^v \frac{dt}{F_1(0, t)} = c^2 \int_0^v \frac{dt}{c^2 - t^2} = \frac{c}{2} \int_0^v \left(\frac{1}{c-t} + \frac{1}{c+t} \right) dt = \frac{c}{2} \log \left(\frac{1 + v/c}{1 - v/c} \right).$$

This is the same as the rescaling function $L(v)$ we met at the beginning, up to a factor of c . Of course if the rescaling function $f(x)$ in Theorem 3.2 is multiplied by a nonzero constant, it has the same relevant properties (except $f'(u) \neq 1$.)

Example 3.8. As a third and final example, consider $F(x, y) = xy$ on $I = (0, \infty)$. This is the set of positive real numbers under multiplication. (The set of all nonzero real numbers is not an interval, and the reader should check to see where the proof of Theorem 3.2 breaks down in this case.) Here $u = 1$ and $F_1(1, x) = x$, so Theorem 3.2 tells us that a rescaling function which converts multiplication on $(0, \infty)$ to addition on \mathbf{R} is

$$f(x) = \int_1^x \frac{dt}{F_1(1, t)} = \int_1^x \frac{dt}{t} = \log x.$$

Of course we already knew that $\log(xy) = \log x + \log y$, but it is still interesting to see how we have rediscovered the logarithm by applying calculus to the general algebraic problem of transforming "addition laws" on intervals into ordinary addition on the real line.

APPENDIX A. HYPERBOLIC FUNCTIONS

The hyperbolic trigonometric functions play the same role relative to $x^2 - y^2 = 1$ that the circular trigonometric functions play relative to $x^2 + y^2 = 1$. We will focus on the hyperbolic analogues of the three basic functions $\sin t$, $\cos t$, and $\tan t$.

Definition A.1. For any real number t , the *hyperbolic sine* and *hyperbolic cosine* of t are the numbers

$$\sinh t = \frac{e^t - e^{-t}}{2}, \quad \cosh t = \frac{e^t + e^{-t}}{2}.$$

The *hyperbolic tangent* of t is the ratio

$$\tanh t = \frac{\sinh t}{\cosh t} = \frac{e^t - e^{-t}}{e^t + e^{-t}}.$$

By some algebra, $\cosh^2 t - \sinh^2 t = 1$. Therefore $(\cosh t, \sinh t)$ lies on the hyperbola $x^2 - y^2 = 1$. This is why the functions are called *hyperbolic*. The hyperbola has two branches, one to the right of the y -axis and the other to the left of the y -axis. Since $\cosh t > 0$ for all t , the point $(\cosh t, \sinh t)$ always lies on the *right* branch of the hyperbola.

Remark A.2. The hyperbolic cosine gives the shape of a freely hanging cable and both $\sinh t$ and $\cosh t$ are used in formulas for isometries of spacetime [1].

As t runs through the real numbers, these hyperbolic functions doesn't behave like their circular analogues at all: they are *not* periodic, $\cosh t$ and $\sinh t$ are *not* bounded (indeed, their values explode when $t \rightarrow \pm\infty$), $\cosh t$ has *no* zeros, and $\tanh t$ has *no* asymptotes. What makes them similar is revealed by calculus: their power series expansions are quite close to those of $\sin t$, $\cos t$, and $\tan t$.

Recall the series for $\sin t$ and $\cos t$:

$$\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots = \sum_{k \geq 0} (-1)^k \frac{t^{2k+1}}{(2k+1)!}, \quad \cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots = \sum_{k \geq 0} (-1)^k \frac{t^{2k}}{(2k)!}.$$

Using the power series for e^t , we can compute the series for $\sinh t$ and $\cosh t$. Starting with

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \cdots, \quad e^{-t} = 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!} + \cdots,$$

we add and subtract and divide in each case by 2:

$$\sinh t = t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots = \sum_{k \geq 0} \frac{t^{2k+1}}{(2k+1)!}, \quad \cosh t = 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots = \sum_{k \geq 0} \frac{t^{2k}}{(2k)!}.$$

These are just like the series for $\sin t$ and $\cos t$ except for the alternating signs. Similarly,

$$\tan t = t + \frac{t^3}{3} + \frac{2t^5}{15} + \frac{17t^7}{315} + \frac{62t^9}{2835} + \cdots, \quad \tanh t = t - \frac{t^3}{3} + \frac{2t^5}{15} - \frac{17t^7}{315} + \frac{62t^9}{2835} - \cdots.$$

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