A "UNIVERSAL" DIVISIBILITY TEST

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The tests for divisibility by 3, 9, and 11 all have a similar flavor: for a positive integer

$$n = a_d 10^d + a_{d-1}10^{d-1} + \cdots + a_1 10 + a_0, \tag{1}$$

where $0 \leq a_i \leq 9$, we have

$$\begin{aligned} n \equiv 0 \bmod 3 &\iff a_d + a_{d-1} + a_{d-2} + \cdots + a_0 \equiv 0 \bmod 3, \\ n \equiv 0 \bmod 9 &\iff a_d + a_{d-1} + a_{d-2} + \cdots + a_0 \equiv 0 \bmod 9, \\ n \equiv 0 \bmod 11 &\iff a_d - a_{d-1} + a_{d-2} - \cdots + (-1)^d a_0 \equiv 0 \bmod 11. \end{aligned}$$

These tests are convenient to use because the sum of the digits of n and the alternating sum of the digits of n are much smaller than n, so we can turn the divisibility problem for n into a divisibility problem for a smaller number. Moreover, we can iterate the test again and again until we are left with a very small number to test.

These three tests generalize to a test for divisibility by *any* integer m relatively prime to 10 (that is, m is not a multiple of 2 or 5). So we will get, for instance, divisibility tests by $7, 13$, and 29. The general test will involve the operation of taking off the units' digit of a positive integer, *e.g.*, turning 1634 into 163 or 78325 into 7832. For $n \geq 1$, let n' be the number that we get after taking off the units' digit of n. So if n is written as in (1),

$$n' = a_d 10^{d-1} + a_{d-1}10^{d-2} + \cdots + a_1 = \frac{n - a_0}{10}. \tag{2}$$

In (2) we removed the digit a_0 and shifted all the other digits into the next lower position (a_1 fills the position previously taken by a_0, and so on).

Here is the universal divisibility test.

Theorem 1. *When* $(m, 10) = 1$*, choose* b *so that* $10b \equiv 1 \bmod m$*. Then*

$$n \equiv 0 \bmod m \iff n' + ba_0 \equiv 0 \bmod m.$$

We will look at a number of examples of this before we discuss the proof.

Example 2. Take $m = 7$. Then $10 \cdot 5 \equiv 1 \bmod 7$, so

$$n \equiv 0 \bmod 7 \iff n' + 5a_0 \equiv 0 \bmod 7. \tag{3}$$

Let's try $n = 11382$. We have $n' = 1138$ and $n' + 5a_0 = 1138 + 5 \cdot 2 = 1148$, so $7|n$ if and only if $7|1148$. Since 1148 is still big, we apply the test again to 1148: $114 + 5 \cdot 8 = 114 + 40 = 154$, so $7|1148$ if and only if $7|154$. Then we replace 154 with $15 + 5 \cdot 4 = 15 + 20 = 35$, which *is* divisible by 7. Thus the original number 11382 is divisible by 7 (because the test is an "if and only if" criterion, so it works in both directions). Explicitly,

$$n = 11382 = 7 \cdot 1626.$$

Let's summarize our successive computations in the following way:

$$11382 \rightsquigarrow 1138 + 5 \cdot 2 = 1148 \rightsquigarrow 114 + 5 \cdot 8 = 154 \rightsquigarrow 15 + 5 \cdot 4 = 35.$$

Any b fitting $10b \equiv 1 \bmod 7$ can be used in place of 5 in this test. Since $10(-2) \equiv 1 \bmod 7$, for instance, we also get a test for divisibility by 7 as

$$n \equiv 0 \bmod 7 \Longleftrightarrow n' - 2a_0 \equiv 0 \bmod 7. \tag{4}$$

This is more convenient to use than (3) since -2 is smaller in magnitude than 5. Of course (3) and (4) are the same test, since $5 \equiv -2 \bmod 7$, but the integers they lead to are different. Let's apply (4) to 11382. The successive numbers we get now are

$$11382 \rightsquigarrow 1138 - 2 \cdot 2 = 1134 \rightsquigarrow 113 - 2 \cdot 4 = 105 \rightsquigarrow 10 - 2 \cdot 5 = 0,$$

which is divisible by 7, so the original number 11382 is divisible by 7.

Example 3. Take $m = 13$. Then $10 \cdot 4 \equiv 1 \bmod 13$, so

$$n \equiv 0 \bmod 13 \Longleftrightarrow n' + 4a_0 \equiv 0 \bmod 13.$$

Taking $n = 11382$ again, the successive numbers under the operation $n \rightsquigarrow n' + 4a_0$ are

$$11382 \rightsquigarrow 1138 + 4 \cdot 2 = 1146 \rightsquigarrow 114 + 4 \cdot 6 = 138 \rightsquigarrow 13 + 4 \cdot 8 = 45 \rightsquigarrow 4 + 4 \cdot 5 = 24,$$

which is *not* divisible by 13, so 11382 is not divisible by 13. (We didn't have to stop there: $24 \rightsquigarrow 2 + 4 \cdot 4 = 18$, which is not divisible by 13.) Trying now $n = 78325$, we compute

$$78325 \rightsquigarrow 7832 + 4 \cdot 5 = 7852 \rightsquigarrow 785 + 4 \cdot 2 = 793 \rightsquigarrow 79 + 4 \cdot 3 = 91 \rightsquigarrow 9 + 4 = 13,$$

so 78325 is divisible by 13. Explicitly,

$$78325 = 13 \cdot 6025.$$

For $m < 50$ with $(10, m) = 1$, Table 1 below lists the inverse of 10 mod m in the second column, using the representative that is smallest in absolute value (so for $m = 7$ we choose -2 rather than 5).

Example 4. From Table 1,

$$n \equiv 0 \bmod 17 \Longleftrightarrow n' - 5a_0 \equiv 0 \bmod 17,$$
$$n \equiv 0 \bmod 19 \Longleftrightarrow n' + 2a_0 \equiv 0 \bmod 19,$$
$$n \equiv 0 \bmod 21 \Longleftrightarrow n' - 2a_0 \equiv 0 \bmod 21,$$
$$n \equiv 0 \bmod 23 \Longleftrightarrow n' + 7a_0 \equiv 0 \bmod 23,$$
$$n \equiv 0 \bmod 27 \Longleftrightarrow n' - 8a_0 \equiv 0 \bmod 27,$$

and

$$n \equiv 0 \bmod 29 \Longleftrightarrow n' + 3a_0 \equiv 0 \bmod 29.$$

Let's see if 1634 is divisible by 29. The operation is $n \rightsquigarrow n' + 3a_0$ in this case, and

$$1634 \rightsquigarrow 163 + 3 \cdot 4 = 175 \rightsquigarrow 17 + 3 \cdot 5 = 32,$$

which is not a multiple of 29, so 1634 is not divisible by 29. Now trying 13108, we get

$$13108 \rightsquigarrow 1310 + 3 \cdot 8 = 1334 \rightsquigarrow 133 + 3 \cdot 4 = 145 \rightsquigarrow 14 + 3 \cdot 5 = 29,$$

which is divisible by 29, so 13108 is divisible by 29. Explicitly,

$$13108 = 29 \cdot 452.$$

Now that we see how Theorem 1 works in practice, let's prove it. The proof will be very short! It depends on writing n as $10n' + a_0$ and doing one multiplication mod m.

m	b
3	1
7	-2
9	1
11	-1
13	4
17	-5
19	2
21	-2
23	7
27	-8
29	3
31	-3
33	10
37	-11
39	4
41	-4
43	13
47	-14
49	5

TABLE 1. A solution to $10b \equiv 1 \bmod m$

Proof. Since $n = 10n' + a_0$,

$$n \equiv 0 \bmod m \Longleftrightarrow 10n' + a_0 \equiv 0 \bmod m.$$

Since 10 mod m is invertible, with inverse b,

$$\begin{aligned} 10n' + a_0 \equiv 0 \bmod m &\Longleftrightarrow b(10n' + a_0) \equiv 0 \bmod m \\ &\Longleftrightarrow n' + ba_0 \equiv 0 \bmod m. \end{aligned}$$

□

All that really happened in the proof is that we divided by 10 working modulo m. If we allow ourselves to use ordinary fractional notation, $10n' + a_0 \equiv 0 \bmod m$ if and only if $n' + a_0/10 \equiv 0 \bmod m$ and the legal form of $1/10 \bmod m$ is $b \bmod m$ since $10b \equiv 1 \bmod m$.

Although we said at the start that the divisibility test in Theorem 1 generalizes the divisibility tests for 3, 9, and 11, which involve adding (or alternately adding and subtracting) all the digits of a number, the usual tests for 3, 9, and 11 don't actually look like the test in Theorem 1. So let's see how Theorem 1 implies the usual tests for 3, 9, and 11. Looking at Table 1, where $b = 1$ for $m = 3$ and 9, and $b = -1$ for $m = 11$, Theorem 1 says

$$\begin{aligned} n \equiv 0 \bmod 3 &\Longleftrightarrow n' + a_0 \equiv 0 \bmod 3, \\ n \equiv 0 \bmod 9 &\Longleftrightarrow n' + a_0 \equiv 0 \bmod 9, \\ n \equiv 0 \bmod 11 &\Longleftrightarrow n' - a_0 \equiv 0 \bmod 11. \end{aligned}$$

Since $10 \equiv 1 \bmod 3$, by (2)

$$n' \equiv a_d + a_{d-1} + \cdots + a_1 \bmod 3,$$

so

$$n' + a_0 \equiv a_d + a_{d-1} + \cdots + a_1 + a_0 \bmod 3.$$

Therefore the test for divisibility by 3 in Theorem 1 is the same as

$$n \equiv 0 \bmod 3 \iff a_d + a_{d-1} + \cdots + a_1 + a_0 \equiv 0 \bmod 3,$$

which is the usual test for divisibility by 3. Since $10 \equiv 1 \bmod 9$, Theorem 1 implies the usual test for divisibility by 9 in the same way. As for 11, since $10 \equiv -1 \bmod 11$ we have

$$n' \equiv a_d(-1)^{d-1} + a_{d-1}(-1)^{d-2} + \cdots + a_1 \bmod 11,$$

so

$$n' - a_0 \equiv a_d(-1)^{d-1} + a_{d-1}(-1)^{d-2} + \cdots + a_1 - a_0 \bmod 11.$$

Therefore Theorem 1 says

$$\begin{aligned} n \equiv 0 \bmod 11 &\iff n' - a_0 \equiv 0 \bmod 11 \\ &\iff a_d(-1)^{d-1} + a_{d-1}(-1)^{d-2} + \cdots + a_1 - a_0 \bmod 11 \\ &\iff (-1)^{d-1}(a_d - a_{d-1} + \cdots + (-1)^{d-1}a_1 + (-1)^d a_0) \equiv 0 \bmod 11 \\ &\iff a_d - a_{d-1} + \cdots + (-1)^{d-1}a_1 + (-1)^d a_0 \equiv 0 \bmod 11, \end{aligned}$$

which is the usual for divisibility by 11.

Remark 5. If we try out the universal divisibility test for m on a number that is too small (relative to m), we may produce larger numbers in the recursion. For example, take $m = 13$ (and $b = 4$). Testing for divisibility of 28 by 13, we get

$$28 \rightsquigarrow 2 + 4 \cdot 8 = 34 \rightsquigarrow 3 + 4 \cdot 4 = 19 \rightsquigarrow 1 + 4 \cdot 9 = 37 \rightsquigarrow 3 + 4 \cdot 7 = 31 \rightsquigarrow 3 + 4 \cdot 1 = 7,$$

which is not divisible by 13 so 28 isn't divisible by 13 either. Notice the sequence went up and down a couple of times before getting very small.

It can also happen that the recursion enters a loop. For example, if we want to test 351 for divisibility by 13 then we get

$$351 \rightsquigarrow 35 + 4 \cdot 1 = 39 \rightsquigarrow 3 + 4 \cdot 9 = 39 \rightsquigarrow 39 \rightsquigarrow 39 \rightsquigarrow \dots.$$

It can be shown that the "universal" test for divisibility by m will lead to rising numbers or a loop only at a stage where the numbers are small relative to m (of size less than $10m$, in fact), at which point you could just stop and do a direct divisibility check.