

# STABLY FREE MODULES

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## 1. INTRODUCTION

Let  $R$  be a commutative ring. When an  $R$ -module has a particular module-theoretic property after direct summing it with a finite free module, it is said to have the property *stably*. For example,  $R$ -modules  $M$  and  $N$  are *stably isomorphic* if  $R^k \oplus M \cong R^k \oplus N$  for some  $k \geq 0$ . An  $R$ -module  $M$  is *stably free* if it is stably isomorphic to a free module:  $R^k \oplus M$  is free for some  $k$ . When  $M$  is finitely generated and stably free, then for some  $k$   $R^k \oplus M$  is finitely generated and free, so  $R^k \oplus M \cong R^\ell$  for some  $\ell$ . Necessarily  $k \leq \ell$  (why?). Are stably isomorphic modules in fact isomorphic? Is a stably free module actually free? Not always, and that's why the concepts are interesting. This "stable mathematics" is part of algebraic  $K$ -theory. Our purpose here is to describe the simplest example of a non-free module which is stably free and then discuss what it means for all stably free modules over a ring to be free.

**Theorem 1.1.** *Let  $R$  be the ring  $\mathbf{R}[x, y, z]/(x^2 + y^2 + z^2 - 1)$ . Let  $T = \{(f, g, h) \in R^3 : xf + yg + zh = 0 \text{ in } R\}$ . Then  $R \oplus T \cong R^3$ , but  $T \not\cong R^2$ .*

The module  $T$  in this theorem is stably free (it is stably isomorphic to  $R^2$ ), but it is not a free module. Indeed, if  $T$  is free then (since  $T$  is finitely generated; the theorem shows it admits a surjection from  $R^3$ ) for some  $n$  we have  $T \cong R^n$ , so  $R \oplus R^n \cong R^3$ . Since  $R^a \cong R^b$  only if  $a = b$ ,  $1 + n = 3$  so  $n = 2$ . But this contradicts the non-isomorphism in the conclusion of the theorem.

It's worth noting that the ranks in the theorem are as small as possible for a non-free stably free module. If  $R$  is any commutative ring and  $M$  is an  $R$ -module such that  $R \oplus M \cong R$  then  $M = 0$ . If  $R \oplus M \cong R^2$  then  $M \cong R$ . The first time we could have  $R \oplus M \cong R^\ell$  with  $M \not\cong R^{\ell-1}$  is  $\ell = 3$ , and Theorem 1.1 shows such an example occurs.

## 2. PROOF OF THEOREM 1.1

In the proof of Theorem 1.1 it will be easy to show  $R \oplus T \cong R^3$ . But the proof that  $T \not\cong R^2$  will require a theorem from topology about vector fields on the sphere. We denote the module as  $T$  because it is related to tangent vectors on the sphere.

*Proof.* Since  $R$  is a ring, on  $R^3$  we can consider the dot product  $R^3 \times R^3 \rightarrow R$ . For example,  $(x, y, z) \cdot (x, y, z) = x^2 + y^2 + z^2 = 1$ . For any  $\mathbf{v} \in R^3$ , let  $r = \mathbf{v} \cdot (x, y, z) \in R$ . Then

$$(\mathbf{v} - r(x, y, z)) \cdot (x, y, z) = \mathbf{v} \cdot (x, y, z) - r(x, y, z) \cdot (x, y, z) = r - r = 0,$$

so  $\mathbf{v} - r(x, y, z) \in T$ . That means  $R^3 = R(x, y, z) + T$ . This sum is direct since  $R(x, y, z) \cap T = (0, 0, 0)$ : if  $r(x, y, z) \in T$  then dotting  $r(x, y, z)$  with  $(x, y, z)$  implies  $r = 0$ . So we have proved

$$(2.1) \quad R^3 = R(x, y, z) \oplus T.$$

Since  $R \cong R(x, y, z)$  by  $r \mapsto r(x, y, z)$ ,  $R^3 \cong R \oplus T$ . Thus  $T$  is stably free.

Now we will show by contradiction that  $T \not\cong R^2$ . Assume  $T \cong R^2$ , so  $T$  has an  $R$ -basis of size 2, say  $(f, g, h)$  and  $(F, G, H)$ . By (2.1) the three vectors  $(x, y, z), (f, g, h), (F, G, H)$  in  $R^3$  are an  $R$ -basis, so the matrix

$$\begin{pmatrix} x & f & F \\ y & g & G \\ z & h & H \end{pmatrix}$$

in  $M_3(R)$  must be invertible: it is the change-of-basis matrix between the standard basis of  $R^3$  and the basis  $(x, y, z), (f, g, h), (F, G, H)$ . Therefore the determinant of this matrix is a unit in  $R$ :

$$(2.2) \quad \det \begin{pmatrix} x & f & F \\ y & g & G \\ z & h & H \end{pmatrix} \in R^\times.$$

It makes sense to evaluate elements of  $R$  at points  $(x_0, y_0, z_0)$  on the unit sphere  $S^2$ : polynomials in  $\mathbf{R}[x, y, z]$  which are congruent modulo  $x^2 + y^2 + z^2 - 1$  take the same value at any  $(x_0, y_0, z_0) \in S^2$  since  $x_0^2 + y_0^2 + z_0^2 - 1 = 0$ . A unit in  $R$  takes nonzero values everywhere on the sphere: if  $a(x, y, z)b(x, y, z) = 1$  in  $R$  then  $a(x_0, y_0, z_0)b(x_0, y_0, z_0) = 1$  in  $\mathbf{R}$  when  $(x_0, y_0, z_0) \in S^2$ . In particular, at each point  $\mathbf{v} \in S^2$  the determinant in (2.2) has a nonzero value, so  $(f(\mathbf{v}), g(\mathbf{v}), h(\mathbf{v})) \in \mathbf{R}^3 - \{\mathbf{0}\}$ . Thus  $\mathbf{v} \mapsto (f(\mathbf{v}), g(\mathbf{v}), h(\mathbf{v}))$  is a nowhere vanishing vector field on  $S^2$  with continuous components (polynomial functions are continuous). But this is impossible: the hairy ball theorem in topology says every continuous vector field on the sphere vanishes at least once.  $\square$

There is a stably free non-free module  $T_{\mathbf{Z}}$  over  $\mathbf{Z}[x, y, z]/(x^2 + y^2 + z^2 - 1)$ . The construction is analogous to the previous one. Elements of  $\mathbf{Z}[x, y, z]/(x^2 + y^2 + z^2 - 1)$  can be evaluated on the real sphere, and the proof that  $T_{\mathbf{Z}}$  is not a free module uses evaluations of polynomials at points on the real sphere as before.

For any  $d \geq 1$ , every continuous vector field on the  $2d$ -dimensional sphere  $S^{2d}$  vanishes somewhere, so over

$$(2.3) \quad R = \mathbf{R}[x_1, \dots, x_{2d+1}]/(x_1^2 + \dots + x_{2d+1}^2 - 1)$$

the tangent module  $T = \{(f_1, \dots, f_{2d+1}) \in R^{2d+1} : \sum x_i f_i = 0 \text{ in } R\}$  is stably free but not free:  $R \oplus T \cong R^{2d+1}$  but  $T \not\cong R^{2d}$ .

### 3. WHEN STABLY FREE MODULES MUST BE FREE

For some rings  $R$ , all stably free finitely generated  $R$ -modules are free. This holds if  $R$  is a field since all vector spaces are free (have bases). It also holds if  $R$  is a PID: a stably free  $R$ -module is a submodule of a finite free  $R$ -module, and any submodule of a finite free module over a PID is a free module. A much more difficult example is when  $R = k[X_1, \dots, X_n]$ , where  $k$  is a field. (This is Serre's conjecture, proved independently by Quillen and Suslin with  $k$  even allowed to be a PID rather than a field.<sup>1</sup>) In this section we show how the task of proving all stably free finitely generated modules over a particular ring  $R$  are free can be formulated as a linear algebra problem over  $R$ . (It is shown in the

<sup>1</sup>The actual problem put forward by Serre was to show any finitely generated projective module over  $k[X_1, \dots, X_n]$  is free. He showed such modules are stably free, so his problem reduces to the version we stated about freeness of stably free finitely generated modules over  $k[X_1, \dots, X_n]$ .

appendix that over any ring, every non-finitely generated module which is stably free is free, so there is no loss of generality in focusing on finitely generated modules.)

To distinguish  $n$ -tuples  $(a_1, \dots, a_n)$  in  $R^n$  from the ideal  $(a_1, \dots, a_n) = Ra_1 + \dots + Ra_n$  in  $R$ , denote the  $n$ -tuple in  $R^n$  as  $[a_1, \dots, a_n]$ .

**Theorem 3.1.** *Fix a nonzero commutative ring  $R$  and a positive integer  $n$ . The following conditions are equivalent.*

- (1) *For any  $R$ -module  $M$ , if  $M \oplus R \cong R^n$  then  $M$  is free.*
- (2) *Every vector  $[a_1, \dots, a_n] \in R^n$  satisfying  $(a_1, \dots, a_n) = R$  is part of a basis of  $R^n$ .*

*Proof.* Both (1) and (2) are true (for all  $R$ ) when  $n = 1$ , so we may suppose  $n \geq 2$ .

(1)  $\Rightarrow$  (2): Suppose  $(a_1, \dots, a_n) = R$ , so  $\sum a_i b_i = 1$  for some  $b_i \in R$ . Set  $\mathbf{a} = [a_1, \dots, a_n]$  and  $\mathbf{b} = [b_1, \dots, b_n]$ . Let  $f: R^n \rightarrow R$  by  $f(\mathbf{v}) = \mathbf{v} \cdot \mathbf{b}$ , so  $f(\mathbf{a}) = 1$  and  $R^n = R\mathbf{a} \oplus \ker f$  by the decomposition

$$\mathbf{v} = f(\mathbf{v})\mathbf{a} + (\mathbf{v} - f(\mathbf{v})\mathbf{a}).$$

(This sum decomposition is unique because if  $\mathbf{v} = r\mathbf{a} + \mathbf{w}$  with  $r \in R$  and  $\mathbf{w} \in \ker f$  then applying  $f$  to both sides shows  $f(\mathbf{v}) = r$ , so  $\mathbf{w} = \mathbf{v} - r\mathbf{a} = \mathbf{v} - f(\mathbf{v})\mathbf{a}$ .) Since  $R\mathbf{a} \cong R$  by  $\mathbf{v} \mapsto \mathbf{v} \cdot \mathbf{b}$  (concretely,  $r\mathbf{a} \mapsto r$ ),  $R^n$  is isomorphic to  $R \oplus \ker f$ , so  $\ker f$  is free by (1). Adjoining  $\mathbf{a}$  to a basis of  $\ker f$  provides us with a basis of  $R^n$ .

(2)  $\Rightarrow$  (1): Let  $g: M \oplus R \rightarrow R^n$  be an  $R$ -module isomorphism. Set  $\mathbf{a} = g(0, 1) = [a_1, \dots, a_n]$ . To show the ideal  $(a_1, \dots, a_n)$  is  $R$ , suppose it is not. Then there is a maximal ideal  $\mathfrak{m}$  containing each  $a_i$ , so  $g(0, 1) \in \mathfrak{m}^n$ . However, the isomorphism  $g$  restricts to an isomorphism from  $\mathfrak{m}(M \oplus R) = \mathfrak{m}M \oplus \mathfrak{m}$  to  $\mathfrak{m}R^n = \mathfrak{m}^n$ , so  $g(0, 1)$  being in  $\mathfrak{m}^n$  implies  $(0, 1) \in \mathfrak{m}M \oplus \mathfrak{m}$ , which is false.

By (2) there is a basis of  $R^n$  containing  $\mathbf{a}$ . Every  $R$ -basis of  $R^n$  contains  $n$  elements<sup>2</sup> so we can write the basis of  $R^n$  as  $\mathbf{v}_1, \dots, \mathbf{v}_n$  with  $\mathbf{v}_1 = \mathbf{a}$ . Then  $g^{-1}(\mathbf{v}_1), \dots, g^{-1}(\mathbf{v}_n)$  is a basis of  $M \oplus R$ , with  $g^{-1}(\mathbf{v}_1) = (0, 1)$ . For  $i = 2, \dots, n$ , write  $g^{-1}(\mathbf{v}_i) = (m_i, c_i)$ . Subtracting a multiple of  $(0, 1)$  from each  $(m_i, c_i)$  for  $i = 2, \dots, n$ , we get a basis  $(0, 1), (m_2, 0), \dots, (m_n, 0)$  of  $M \oplus R$ . Writing any  $(m, 0)$  as a linear combination of these shows  $m_2, \dots, m_n$  spans  $M$  as an  $R$ -module and is linearly independent, so  $M$  is free.  $\square$

**Corollary 3.2.** *For a commutative ring  $R$ , the following conditions are equivalent.*

- (1) *For all  $R$ -modules  $M$ , if  $M \oplus R \cong R^n$  for some  $n$  then  $M$  is free.*
- (2) *For all  $n \geq 1$ , every vector  $[a_1, \dots, a_n] \in R^n$  satisfying  $(a_1, \dots, a_n) = R$  is part of a basis of  $R^n$ .*
- (3) *All stably free finitely generated  $R$ -modules are free.*

*Proof.* (1)  $\Leftrightarrow$  (2): This equivalence is Theorem 3.1 for all  $n$ .

(1)  $\Rightarrow$  (3): Suppose  $M$  is a stably free  $R$ -module, so  $M \oplus R^k \cong R^\ell$  for some  $k$  and  $\ell$ . We want to show  $M$  is free. If  $k = 0$  then obviously  $M$  is free. If  $k \geq 1$  then  $(M \oplus R^{k-1}) \oplus R \cong R^\ell$ , so (1) with  $n = \ell$  tells us that  $M \oplus R^{k-1}$  is free. By induction on  $k$ , the module  $M$  is free.

(3)  $\Rightarrow$  (1): If  $M \oplus R \cong R^n$  for some  $n$  then  $M$  is stably free, and thus  $M$  is free by (3).  $\square$

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<sup>2</sup>If  $R^n \cong R^{n'}$  as  $R$ -modules then, for any maximal ideal  $\mathfrak{m}$  in  $R$ , that  $R$ -module isomorphism implies  $\mathfrak{m}^n \cong \mathfrak{m}^{n'}$  and therefore  $R^n/\mathfrak{m}^n \cong R^{n'}/\mathfrak{m}^{n'}$ , so  $(R/\mathfrak{m})^n \cong (R/\mathfrak{m})^{n'}$  as  $R/\mathfrak{m}$ -vector spaces. By linear algebra over the field  $R/\mathfrak{m}$ ,  $n = n'$ .

Corollary 3.2(2) expresses the freeness of all stably free finitely generated  $R$ -modules as a problem in linear algebra in  $R^n$  (over all  $n$ ). The condition there that the coordinates generate the unit ideal is necessary if  $[a_1, \dots, a_n]$  has a chance to be part of a basis of  $R^n$ :

**Theorem 3.3.** *If  $[a_1, \dots, a_n] \in R^n$  is part of a basis of  $R^n$  then the ideal  $(a_1, \dots, a_n)$  is the unit ideal.*

*Proof.* We are assuming there is an  $R$ -basis of  $R^n$  that contains the  $n$ -tuple  $[a_1, \dots, a_n]$ . Any basis has  $n$  elements, so write the basis as  $\mathbf{v}_1, \dots, \mathbf{v}_n$  with  $\mathbf{v}_1 = [a_1, \dots, a_n]$ . Write each  $\mathbf{v}_j$  in coordinates relative to the standard basis of  $R^n$ , say  $\mathbf{v}_j = [c_{1j}, \dots, c_{nj}]$  (so  $a_i = c_{i1}$ ). Then the matrix  $(c_{ij})$  has the  $\mathbf{v}_j$ 's as its columns, so this matrix describes the linear transformation  $R^n \rightarrow R^n$  sending the standard basis of  $R^n$  to  $\mathbf{v}_1, \dots, \mathbf{v}_n$ . Since the  $\mathbf{v}_j$ 's form a basis, this matrix is invertible:  $\det(c_{ij}) \in R^\times$ . Expanding the determinant along its first column shows  $\det(c_{ij})$  is an  $R$ -linear combination of  $a_1, \dots, a_n$ , so  $\det(c_{ij}) \in (a_1, \dots, a_n)$ . Therefore the ideal  $(a_1, \dots, a_n)$  contains a unit, so the ideal is  $R$ .  $\square$

Thus all stably free finitely generated  $R$ -modules are free if and only if for all  $n$  the “obvious” necessary condition for a vector in  $R^n$  to be part of a basis of  $R^n$  is a sufficient condition.

**Example 3.4.** If  $[a_1, \dots, a_n]$  in  $\mathbf{Z}^n$  is part of a basis of  $\mathbf{Z}^n$  then  $\gcd(a_1, \dots, a_n) = 1$ . For example, the vector  $[6, 9, 15]$  is not part of a basis of  $\mathbf{Z}^3$  since its coordinates are all multiples of 3. The vector  $[6, 10, 15]$  has no common factors among its coordinates (although each pair of coordinates has a common factor). Is it part of a basis of  $\mathbf{Z}^3$ ? Essentially we are asking if the necessary condition in Theorem 3.3 is also sufficient over  $\mathbf{Z}$ . It is in this case: the vectors  $[6, 10, 15]$ ,  $[1, 1, 0]$ , and  $[0, 3, 11]$  are a basis of  $\mathbf{Z}^3$ . (A matrix with these vectors as the columns has determinant  $\pm 1$ .)

For any  $R$ , the necessary condition  $(a_1, \dots, a_n) = R$  in Theorem 3.3 is actually sufficient for  $[a_1, \dots, a_n]$  to be part of a basis of  $R^n$  when  $n = 1$  and  $n = 2$ . For  $n = 1$ , if  $(a_1) = R$  then  $a_1$  is a unit and thus is a basis of  $R$  as an  $R$ -module. For  $n = 2$ , if  $(a_1, a_2) = R$  then there are  $b_1, b_2 \in R$  such that  $a_1 b_1 + a_2 b_2 = 1$ , so the matrix  $\begin{pmatrix} a_1 & -b_2 \\ a_2 & b_1 \end{pmatrix}$  has determinant 1 and therefore its columns are a basis of  $R^2$ . What if  $n > 2$ ? The necessary condition is sufficient when  $R$  is a PID by Corollary 3.2 since we already saw that stably free  $\Rightarrow$  free when  $R$  is a PID. (e.g.,  $R = \mathbf{Z}$  or  $F[X]$ ). More generally, the necessary condition is sufficient when  $R$  is a Dedekind domain [6], [7], but Theorem 1.1 provides us with a ring admitting a stably free module that is not free, and this leads to a counterexample when  $n = 3$ .

**Example 3.5.** Let  $R = \mathbf{R}[x, y, z]/(x^2 + y^2 + z^2 - 1)$ . In the free module  $R^3$ , the triple  $[x, y, z]$  satisfies the condition of Theorem 3.3: the ideal  $(x, y, z)$  of  $R$  is the unit ideal since  $x^2 + y^2 + z^2 = 1$  in  $R$ . However, there is no basis of  $R^3$  containing  $[x, y, z]$ . Indeed, assume there is a basis  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  where  $\mathbf{v}_1 = [x, y, z]$ . Then there is a matrix in  $\mathrm{GL}_3(R)$  with first column  $\mathbf{v}_1$ , and the argument in the proof of Theorem 1.1 derives a contradiction from this.

The “sphere rings”  $\mathbf{R}[x_1, \dots, x_n]/(x_1^2 + \dots + x_n^2 - 1)$  for any odd  $n \geq 3$  provide additional examples where the condition in Theorem 3.3 is not sufficient to guarantee an  $n$ -tuple in  $R^n$  is part of a basis of  $R^n$ .

Another use of the ring  $R = \mathbf{R}[x, y, z]/(x^2 + y^2 + z^2 - 1)$  as a counterexample in algebra involves matrices with trace 0. Since  $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ , any matrix of the form  $AB - BA$  (called a commutator) has trace 0. Over a field, the converse holds [1]: every square matrix with trace 0 is a commutator. However, the matrix  $\begin{pmatrix} x & y \\ z & -x \end{pmatrix}$  in  $M_2(R)$  has trace 0 and it is

proved in [8] that this matrix is not a commutator in  $M_2(R)$ . But all is not lost. A matrix with trace 0 is always a sum of two commutators [5].

#### APPENDIX A. THEOREMS OF GABEL AND BASS

Our discussion of stably free modules focused on finitely generated ones. The reason is that in the case of non-finitely generated modules there are no interesting stably free modules.

**Theorem A.1** (Gabel). *If  $M$  is stably free and not finitely generated then  $M$  is free.*

*Proof.* Let  $F := M \oplus R^k$  be free. We want to show  $M$  is free.

Projection from  $F$  onto  $M$  is a surjective linear map, so  $M$  not being finitely generated implies  $F$  is not finitely generated. Let  $\{e_i\}_{i \in I}$  be a basis of  $F$ , so the index set  $I$  is infinite.

Projection from  $F$  onto  $R^k$  is a surjective linear map  $f: F \rightarrow R^k$  with kernel  $M$ . The standard basis of  $R^k$  is in the image of the span of finitely many  $e_i$ 's, say the submodule  $F' := Re_1 + \dots + Re_\ell$  has  $f(F') = R^k$ . For any  $\mathbf{v} \in F$ ,  $f(\mathbf{v}) = f(\mathbf{v}')$  for some  $\mathbf{v}' \in F'$ . Then  $\mathbf{v} - \mathbf{v}' \in \ker f = M$ , so  $F = F' + M$ . The module  $F'$  is finite free and  $F'/(M \cap F') \cong R^k$ . Since  $R^k$  is free (and thus a projective module), there is an isomorphism  $F' \cong N \oplus R^k$  where  $N = M \cap F'$ . Since  $F' + M = F$ ,  $F/F' \cong M/N$  and  $F/F' = \bigoplus_{i > \ell} Re_i$  is free with *infinite* rank, so we can write  $F/F' \cong R^k \oplus F''$  for some free  $F''$ . Therefore  $M/N$  is free, so

$$M \cong N \oplus (M/N) \cong N \oplus (F/F') \cong N \oplus R^k \oplus F'' \cong F' \oplus F'',$$

which is free.  $\square$

To prove all stably free modules over a (nonzero) ring  $R$  are free is the same as showing  $M \oplus R^k \cong R^\ell \Rightarrow M$  is free for any  $k$  and  $\ell$ . When such an isomorphism occurs,  $\ell - k = \dim_{R/\mathfrak{m}}(M/\mathfrak{m}M)$  for all maximal ideals  $\mathfrak{m}$  in  $R$ , so  $\ell - k$  is well-defined by  $M$  although  $\ell$  and  $k$  are not. We call  $\ell - k$  the *rank* of  $M$ . For example, if  $M \oplus R \cong R^n$  then  $M$  has rank  $n - 1$ . We will prove a theorem of Bass which reduces the verification that all stably free  $R$ -modules are free to the case of even rank.

**Lemma A.2.** *If  $M \oplus R \cong R^{2d}$  for some  $d \geq 1$  then  $M \cong R \oplus N$  for some  $R$ -module  $N$ .*

*Proof.* Composing an isomorphism  $R^{2d} \cong M \oplus R$  with projection to the second summand gives us a surjective map  $\varphi: R^{2d} \rightarrow R$  with kernel isomorphic to  $M$ . Since any linear map  $R^{2d} \rightarrow R$  is dotting with a fixed vector, there is some  $\mathbf{w} \in R^{2d}$  such that  $\varphi(\mathbf{v}) = \mathbf{v} \cdot \mathbf{w}$  for all  $\mathbf{v} \in R^{2d}$ . Set  $\mathbf{w} = (c_1, \dots, c_{2d})$ . Then

$$\varphi(c_2, -c_1, \dots, c_{2d}, -c_{2d-1}) = 0.$$

Let  $\mathbf{u} = (c_2, -c_1, \dots, c_{2d}, -c_{2d-1}) \in \ker \varphi \cong M$ . We will show there is a submodule  $N$  of  $M$  such that  $M \cong R \oplus N$ .

Choose  $(r_1, \dots, r_{2d}) \in R^{2d}$  such that  $\varphi(r_1, \dots, r_{2d}) = 1$ , so  $c_1 r_1 + \dots + c_{2d} r_{2d} = 1$ . Then  $\mathbf{u} \cdot (r_2, -r_1, \dots, r_{2d}, -r_{2d-1}) = 1$ , so the linear map  $f: R^{2d} \rightarrow R$  given by  $f(\mathbf{v}) = \mathbf{v} \cdot (r_2, -r_1, \dots, r_{2d}, -r_{2d-1})$  satisfies  $f(\mathbf{u}) = 1$ . Since  $\mathbf{u} \in \ker \varphi$ , the restriction of  $f$  to a linear map  $\ker \varphi \rightarrow R$  is surjective and restricts to an isomorphism  $R\mathbf{u} \rightarrow R$ . Thus  $M \cong \ker \varphi = R\mathbf{u} \oplus \ker f \cong R \oplus \ker f$ .  $\square$

**Remark A.3.** It is generally false that if  $M \oplus R \cong R^{2d+1}$  then  $M \cong R \oplus N$  for some  $N$ . An example is  $R$  being the sphere ring (2.3) when  $2d + 1 \neq 1, 3$  or  $7$  and  $M$  being the tangent module  $T$ . Our work in Section 2 shows  $T \oplus R \cong R^{2d+1}$ . A proof that  $T \not\cong R \oplus N$  for any  $N$  is in [3, pp. 33–35].

**Theorem A.4** (Bass). *The following conditions on a commutative ring  $R$  are equivalent.*

- (1) *All stably free finitely generated  $R$ -modules are free.*
- (2) *All stably free finitely generated  $R$ -modules of even rank are free.*

*Proof.* It's clear that (1)  $\Rightarrow$  (2). To show (2)  $\Rightarrow$  (1), suppose  $M \oplus R^k \cong R^\ell$  (so  $k \leq \ell$ ) with  $\ell - k$  an odd number. If  $k = 0$  then  $M$  is free. If  $k > 0$  then  $(M \oplus R) \oplus R^{k-1} \cong R^\ell$ , so  $M \oplus R$  is stably free of even rank  $\ell - (k - 1)$ . Then  $M \oplus R \cong R^{\ell-k+1}$ , so  $M \cong R \oplus N$  for some  $N$  by Lemma A.2. Therefore  $N \oplus R^2$  is free of even rank  $\ell - k + 1$ , so  $N$  is stably free of odd rank  $(\ell - k + 1) - 2 = \ell - k - 1$ . By induction  $N \cong R^{\ell-k-1}$ , so  $M \cong R \oplus N \cong R^{\ell-k}$ .  $\square$

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