Throughout this note, we work with linear operators on complex vector spaces, of finite dimension. Any such operator has an eigenvector, by the fundamental theorem of algebra.

**Theorem 1.** If $A_1, \ldots, A_r$ are commuting linear operators, they have a common eigenvector.

**Proof.** We induct on $r$, the result being clear if $r = 1$ since we work over the complex numbers: every (complex) linear operator has an eigenvector.

Now assume $r \geq 2$. Let $A_r$ have an eigenvalue $\lambda \in \mathbb{C}$, and let $E_\lambda = \{v : A_r v = \lambda v\}$ be the $\lambda$-eigenspace for $A_r$. For $v \in E_\lambda$, $A_r(A_i v) = A_i(A_r v) = A_i(\lambda v) = \lambda (A_i v)$, so $A_i v \in E_\lambda$. Thus each $A_i$ restricts to a linear operator on $E_\lambda$.

Note $A_1|_{E_\lambda}, \ldots, A_{r-1}|_{E_\lambda}$ are $r-1$ commuting linear operators on the finite-dimensional space $E_\lambda$. By induction, they have a common eigenvector in $E_\lambda$. It is also an eigenvector for $A_r$. We’re done. □

Of course a common eigenvector for $A_1, \ldots, A_r$ will not usually have a common eigenvalue.

**Theorem 2.** If $A_1, \ldots, A_r$ are commuting linear operators, and each one is diagonalizable, they are simultaneously diagonalizable, i.e., there is a basis of simultaneous eigenvectors for the $A_i$.

**Proof.** This is essentially the same type of argument as in Theorem 1, but the stronger hypothesis (commutativity and individual diagonalizability) allows for a stronger conclusion (basis of simultaneous eigenvectors, not only one simultaneous eigenvector). The result is clear if $r = 1$, so assume $r \geq 2$. Let $E_\lambda$ be an eigenspace for $A_r$, so as before, $A_i(E_\lambda) \subset E_\lambda$. By induction on $r$ (this is the number of operators, not the dimension of the space), there is a basis for $E_\lambda$ consisting of simultaneous eigenvectors for $A_1|_{E_\lambda}, \ldots, A_{r-1}|_{E_\lambda}$, and of course they are eigenvectors for $A_r|_{E_\lambda}$. Since $\lambda$ was any eigenvalue for $A_r$, we see each eigenspace of $A_r$ contains a basis of simultaneous eigenvectors for $A_1, \ldots, A_r$. By hypothesis, the whole space on which $A_r$ acts is the direct sum of the eigenspaces for $A_r$, so we are done. □

Note that Theorems 1 and 2 did not need the operators to be invertible.