SIMULTANEOUS COMMUTATIVITY OF OPERATORS

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Throughout this note, we work with linear operators on *complex* vector spaces, of finite dimension. Any such operator has an eigenvector, by the fundamental theorem of algebra.

Theorem 1. *If $A_1, \ldots, A_r$ are commuting linear operators, they have a common eigenvector.*

Proof. We induct on r, the result being clear if $r = 1$ since we work over the complex numbers: every (complex) linear operator has an eigenvector.

Now assume $r \geq 2$.

Let A_r have an eigenvalue $\lambda \in \mathbf{C}$, and let

$$E_\lambda = \{v : A_r v = \lambda v\}$$

be the λ-eigenspace for A_r. For $v \in E_\lambda$, $A_r(A_i v) = A_i(A_r v) = A_i(\lambda v) = \lambda(A_i v)$, so $A_i v \in E_\lambda$. Thus each A_i restricts to a linear operator on E_λ.

Note $A_1|_{E_\lambda}, \ldots, A_{r-1}|_{E_\lambda}$ are $r-1$ commuting linear operators on the finite-dimensional space E_λ. By induction, they have a common eigenvector in E_λ. It is also an eigenvector for A_r. We're done. $\square$

Of course a common eigenvector for $A_1, \ldots, A_r$ will not usually have a common eigenvalue.

Theorem 2. *If $A_1, \ldots, A_r$ are commuting linear operators, and each one is diagonalizable, they are simultaneously diagonalizable, i.e., there is a basis of simultaneous eigenvectors for the A_i.*

Proof. This is essentially the same type of argument as in Theorem 1, but the stronger hypothesis (commutativity and individual diagonalizability) allows for a stronger conclusion (basis of simultaneous eigenvectors, not only one simultaneous eigenvector). The result is clear if $r = 1$, so assume $r \geq 2$. Let E_λ be an eigenspace for A_r, so as before, $A_i(E_\lambda) \subset E_\lambda$. By induction on r (this is the number of operators, not the dimension of the space), there is a basis for E_λ consisting of simultaneous eigenvectors for $A_1|_{E_\lambda}, \ldots, A_{r-1}|_{E_\lambda}$, and of course they are eigenvectors for $A_r|_{E_\lambda}$. Since λ was any eigenvalue for A_r, we see each eigenspace of A_r contains a basis of simultaneous eigenvectors for $A_1, \ldots, A_r$. By hypothesis, the whole space on which A_r acts is the direct sum of the eigenspaces for A_r, so we are done. $\square$

Note that Theorems 1 and 2 did not need the operators to be invertible.