

NOETHERIAN MODULES

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1. INTRODUCTION

In a finite-dimensional vector space, every subspace is finite-dimensional and the dimension of a subspace is at most the dimension of the whole space. Unfortunately, the naive analogue of this for modules and submodules is wrong:

- (1) A submodule of a finitely generated module need not be finitely generated.
- (2) Even if a submodule of a finitely generated module is finitely generated, the minimal number of generators of the submodule is not bounded above by the minimal number of generators of the original module.

Example 1.1. Every commutative ring R is finitely generated as an R -module, namely with the generator 1, and the submodules of R are its ideals. Therefore a commutative ring that has an ideal that is not finitely generated gives us an example of a finitely generated module and non-finitely generated submodule.

Let $R = \mathbf{R}[X_1, X_2, \dots]$ be the polynomial ring over \mathbf{R} (or any field) in countably many variables. Inside R let $I = (X_1, X_2, \dots)$ be the ideal generated by the variables: this is the set of polynomials with constant term 0. To prove I is not finitely generated as an R -module, we will show any finitely generated ideal $Rf_1 + Rf_2 + \dots + Rf_k$ in R doesn't contain any X_i for all large i , so this ideal is not I .

Since each of the polynomials f_1, \dots, f_k involves only a finite number of variables, there's a large n such that all X_i appearing in one of f_1, \dots, f_k have $i < n$. The substitution homomorphism $R \rightarrow \mathbf{R}$ that sends X_i to 0 for $i < n$ and X_i to 1 for $i \geq n$ sends f_1, \dots, f_k to 0 and therefore it sends every R -linear combination of f_1, \dots, f_k to 0. Since this homomorphism sends X_i to 1 for $i \geq n$, such X_i do not lie in $Rf_1 + \dots + Rf_k$.

Example 1.2. Here is an interesting example from complex analysis. Let R be the ring of entire functions on \mathbf{C} , *i.e.*, R consists of power series with complex coefficients and infinite radius of convergence. It turns out that every finitely generated ideal in R is a principal ideal, but that does not mean all ideals in R are principal: one example of an ideal in R that is not finitely generated is the ideal of entire functions vanishing on all but finitely many integers (the integers where the function doesn't vanish can vary with the function). Proofs of these facts about R require hard theorems in complex analysis about the existence of holomorphic functions with prescribed zeros. See [1, Remark 3.5.4, Corollary 3.5.8].

Example 1.3. An example of a finitely generated module and finitely generated submodule requiring more generators than the larger module is $R = \mathbf{Z}[X]$ and $I = (2, X)$. As an R -module, R requires the single generator 1. The ideal I is not principal, so the fewest number of generators needed for I as an R -module is 2.

Another example is $R = \mathbf{R}[X, Y]$ and $I = (X, Y)$ since I is a non-principal ideal in R .

The property of being finitely generated is not well-behaved on passage to submodules (that is, a finitely generated module can have non-finitely generated submodules), so we

will give a name to the modules in which every submodule is finitely generated. Emmy Noether was the first mathematician to make a systematic study of this property, so these modules are named after her.

Definition 1.4. Let R be a commutative ring. An R -module is called *Noetherian* if every submodule is finitely generated.

The significance of the Noetherian condition on modules is twofold: (1) lots of modules that arise in algebra satisfy this condition and (2) this condition behaves well under many standard constructions on modules. Imposing the Noetherian condition on modules in a theorem is often regarded as a rather mild restriction.

Example 1.5. If F is a field, a finite-dimensional F -vector space V is a Noetherian F -module, since the submodules of V are its subspaces and they are all finite-dimensional by standard linear algebra.

Example 1.6. Generalizing the previous example, if R is a PID then every finitely generated R -module is a Noetherian module. This will be a consequence of Theorem 2.6.

An example of a non-Noetherian module is any module that is not finitely generated. For example, an infinite-dimensional vector space over a field F is a non-Noetherian F -module, and for any nonzero ring R the countable direct sum $\bigoplus_{n \geq 1} R$ is a non-Noetherian R -module. If a ring R has an ideal that is not finitely generated then R is a non-Noetherian R -module.

The next theorem gives some standard equivalent conditions for being a Noetherian module. We will not use these other conditions here. Condition (2), called the ascending chain condition (ACC) can be turned around into a descending chain condition that defines the class of Artinian modules, and condition (3) leads to the idea of “Noetherian induction”, which is useful in algebraic geometry.

Theorem 1.7. *The following conditions on an R -module M are equivalent:*

- (1) *all submodules of M are finitely generated (i.e., M is a Noetherian R -module).*
- (2) *any infinite increasing sequence of submodules $N_1 \subset N_2 \subset N_3 \subset \dots$ in M eventually stabilizes: $N_k = N_{k+1}$ for all large k .¹*
- (3) *Every nonempty collection S of submodules of M contains a maximal element with respect to inclusion: there’s a submodule in S not strictly contained in any other submodule in S .*

Proof. (1) \Rightarrow (2): If $N_1 \subset N_2 \subset \dots$ is an increasing sequence of submodules, let $N = \bigcup_{i \geq 1} N_i$. This is a submodule since any pair of elements in N lies in a common N_i , by the increasing condition, so N is closed under addition and multiplication by elements of R . By (1), N is finitely generated. Using the increasing condition again, any finite subset of N lies in a common N_i , so a finite generating set of N is in some N_i . Thus $N \subset N_i$, and of course also $N_i \subset N$, so $N = N_i$. Then for all $j \geq i$, $N_i \subset N_j \subset N = N_i$, so $N_j = N_i$.

(2) \Rightarrow (3): We will prove the negation of (3) implies the negation of (2). If (3) is false then there is a nonempty collection S of submodules of M containing no maximal member with respect to inclusion. Therefore if we start with any module N_1 in S , we can recursively find modules N_2, N_3, \dots such that N_i strictly contains N_{i-1} for all $i \geq 2$. (If there were no such N_i then N_{i-1} would be a maximal element of S , which doesn’t exist.)

¹The notation \subset only means containment, not strict containment.

(3) \Rightarrow (1): Let N be a submodule of M . To prove N is finitely generated, let S be the set of all finitely generated submodules of N . By (3), there is an $\tilde{N} \in S$ that's contained in no other element of S , so \tilde{N} is a finitely generated submodule of N and no other finitely generated submodule of N contains \tilde{N} . We will show $\tilde{N} = N$ by contradiction, which would prove N is finitely generated. If $\tilde{N} \neq N$, pick $n \in N - \tilde{N}$. Since \tilde{N} is finitely generated, also $\tilde{N} + Rn$ is finitely generated, so $\tilde{N} + Rn \in S$. However, $\tilde{N} + Rn$ strictly contains \tilde{N} , which contradicts maximality of \tilde{N} as a member of S . Thus $\tilde{N} = N$. □

2. PROPERTIES OF NOETHERIAN MODULES

Theorem 2.1. *If M is a Noetherian R -module then every submodule of M is Noetherian.*

Proof. This is an immediate consequence of the definition of a Noetherian module, since a submodule of a submodule is a submodule. □

Theorem 2.2. *If M is a Noetherian R -module then every quotient module M/N is Noetherian.*

Proof. Every submodule of M/N has the form L/N where L is a submodule of M with $N \subset L \subset M$. Since M is Noetherian, L is finitely generated, and the reduction of those generators mod N will generate L/N as an R -module. □

Theorem 2.3. *Let M be an R -module and N be a submodule. Then M is Noetherian if and only if N and M/N are Noetherian.*

Proof. If M is Noetherian then N and M/N are Noetherian by Theorems 2.1 and 2.2. Conversely, suppose N and M/N are Noetherian. To prove M is Noetherian, let L be a submodule of M . Then the image of L in M/N is finitely generated and $L \cap N$ is finitely generated. Let $x_1, \dots, x_k \in L$ generate the image of L in M/N and let y_1, \dots, y_ℓ generate $L \cap N$. For any $x \in L$, we have $x \equiv r_1x_1 + \dots + r_kx_k \pmod{N}$ for some $r_i \in R$, so $x - \sum r_ix_i \in L \cap N$. Therefore $x - \sum r_ix_i = \sum s_jy_j$ with $s_j \in R$, so $x = \sum r_ix_i + \sum s_jy_j$. Therefore L is spanned by $x_1, \dots, x_k, y_1, \dots, y_\ell$. □

Make sure to remember the ideas in this proof, as it's the only property of Noetherian modules we discuss here that involves a real idea (how to pass from a property of submodules of N and M/N to that property for submodules of M).

Theorem 2.4. *If M and N are Noetherian R -modules then their direct sum $M \oplus N$ is a Noetherian R -module.*

Proof. A fake proof would say that a submodule of $M \oplus N$ has the form $P \oplus Q$ for submodules P of M and Q of N , so P and Q are each finitely generated, and the union of those generating sets is a finite generating set for $P \oplus Q$. The reason this proof is fake is that submodules of $M \oplus N$ need not be of the form $P \oplus Q$. For example, inside $\mathbf{Z} \oplus \mathbf{Z}$ is the \mathbf{Z} -submodule $\mathbf{Z}(1, 1) = \{(n, n) : n \in \mathbf{Z}\}$.

For a valid proof, apply Theorem 2.3 to the module $M \oplus N$ and submodule $M \oplus 0 \cong M$, where $(M \oplus N)/(M \oplus 0) \cong N$. □

Corollary 2.5. *If M_1, \dots, M_k are Noetherian R -modules then $M_1 \oplus \dots \oplus M_k$ is a Noetherian R -module.*

Proof. Induct on k , with $k = 2$ being Theorem 2.4. □

Theorem 2.6. *If R is a PID then every finitely generated R -module is a Noetherian R -module.*

Proof. Let M be a finitely generated R -module with generators m_1, \dots, m_k . Then there is a surjective R -linear map $f: R^k \rightarrow M$ by $f(c_1, \dots, c_k) = c_1 m_1 + \dots + c_k m_k$, so M is isomorphic to a quotient module of R^k . Since R is a PID it is a Noetherian R -module, and therefore so is the k -fold direct sum R^k (Theorem 2.5) and so is any quotient module of R^k (Theorem 2.2). \square

Remark 2.7. When R is a PID, the number of generators in a finitely generated R -module behaves like vector spaces: if M is a module over a PID with n generators then every submodule of M needs at most n generators. We don't discuss a proof here.

3. NOETHERIAN RINGS

Definition 3.1. A commutative ring R is called *Noetherian* if all ideals of R are finitely generated.

A simple (and boring) example of a Noetherian ring is a field. A more general class of examples are PIDs, since all of their ideals are singly generated. Noetherian rings can be regarded as a good generalization of PIDs: the property of all ideals being singly generated is often not preserved under common ring-theoretic constructions (*e.g.*, \mathbf{Z} is a PID but $\mathbf{Z}[X]$ is not), but the property of all ideals being finitely generated does remain valid under many constructions of new rings from old rings. For example, we will see below that every quadratic ring $\mathbf{Z}[\sqrt{d}]$ is Noetherian, even though many of these rings are not PIDs.

The standard example of a non-Noetherian ring is a polynomial ring $K[X_1, X_2, \dots]$ in infinitely many variables over a field K . The ring of entire functions on \mathbf{C} is also non-Noetherian since it has an ideal that is not finitely generated (Example 1.2). So that you don't think non-Noetherian rings must always supposed to be "really huge", there is a non-Noetherian ring contained in $\mathbf{Q}[X]$: the ring of integral-valued polynomials

$$\text{Int}(\mathbf{Z}) = \{f \in \mathbf{Q}[X] : f(\mathbf{Z}) \subset \mathbf{Z}\}$$

is not Noetherian. This ring is bigger than $\mathbf{Z}[X]$, *e.g.*, $\binom{X}{2} = \frac{X(X-1)}{2}$ is in $\text{Int}(\mathbf{Z}) - \mathbf{Z}[X]$, as is $\binom{X}{n} = \frac{X(X-1)\dots(X-n+1)}{n!}$ for any $n \geq 2$.

The equivalent conditions for being a Noetherian module in Theorem 1.7 carry over to conditions for being a Noetherian ring:

Theorem 3.2. *The following conditions on a commutative ring R are equivalent:*

- (1) *R is Noetherian: all ideals of R are finitely generated.*
- (2) *any infinite increasing sequence of ideals $I_1 \subset I_2 \subset I_3 \subset \dots$ in R eventually stabilizes: $I_k = I_{k+1}$ for all large k .*
- (3) *Every nonempty collection S of ideals of R contains a maximal element with respect to inclusion: there's an ideal in S not strictly contained in any other ideal in S .*

The following two theorems put the second condition of Theorem 3.2 to use.

Theorem 3.3. *Let R be a Noetherian ring. Any surjective ring homomorphism $R \rightarrow R$ is injective, and thus is an isomorphism.*

Proof. Let $\varphi: R \rightarrow R$ be a surjective ring homomorphism. Each iterate φ^n (the n -fold composition of φ with itself) is surjective and the kernels $K_n = \ker(\varphi^n)$ are ideals that form an increasing chain:

$$K_1 \subset K_2 \subset K_3 \subset \cdots .$$

Since R is Noetherian, $K_n = K_{n+1}$ for some n . Pick $y \in \ker \varphi$, so $\varphi(y) = 0$. Since φ^n is surjective, $y = \varphi^n(x)$, so $0 = \varphi(\varphi^n(x)) = \varphi^{n+1}(x)$. Then $x \in \ker(\varphi^{n+1}) = \ker(\varphi^n)$, so $y = \varphi^n(x) = 0$. Thus $\ker \varphi$ is 0, so φ is injective. \square

There is no analogue of Theorem 3.3 for injective ring homomorphisms. For example, $\mathbf{R}[X]$ is a Noetherian ring since it's a PID and the substitution homomorphism $f(X) \mapsto f(X^2)$ on $\mathbf{R}[X]$ is an injective ring homomorphism that is not surjective.

Theorem 3.4. *If R is a Noetherian integral domain that is not a field, then every nonunit in R can be factored into irreducibles.*

We assume R is not a field because irreducible factorizations don't have any meaning for units, so we want R to contain some nonzero elements that aren't units.

Proof. This will be a massive proof by contradiction.

Suppose there is an element a in R that is not 0 or a unit and has no irreducible factorization. We will find another nonzero nonunit $b \in R$ that does not admit a factorization into irreducibles and such that there is a strict inclusion of ideals $(a) \subset (b)$.

Since a is not irreducible and it is not 0 nor a unit, there is a factorization $a = bc$ where b and c are nonunits (and obviously they are not 0 either). If both b and c have an irreducible factorization then so does a (just multiply together irreducible factorizations for b and c), so at least one of b or c has no irreducible factorization. Without loss of generality, say b has no irreducible factorization. Then since c is not a unit, the inclusion $(a) \subset (b)$ is strict.

Rewriting b as a_1 , we have a strict containment of ideals

$$(a) \subset (a_1)$$

where a_1 is a nonzero nonunit with no irreducible factorization. Using a_1 in the role of a in the previous paragraph, there is a strict inclusion of ideals

$$(a_1) \subset (a_2)$$

for some nonzero nonunit a_2 that has no irreducible factorization. This argument can be repeated and leads to an infinite increasing chain of (principal) ideals

$$(3.1) \quad (a) \subset (a_1) \subset (a_2) \subset (a_3) \subset \cdots$$

where all inclusions are strict. This is impossible in a Noetherian ring, so we have a contradiction. Therefore nonzero nonunits without an irreducible factorization do not exist: all nonzero nonunits in R have an irreducible factorization. \square

This theorem is *not* saying a Noetherian integral domain has unique factorization: just because elements have irreducible factorizations doesn't mean those are unique (up to the order of multiplication and multiplication of terms by units). In fact many Noetherian integral domains do not have unique factorization.

For the rest of this section we want to show that some basic operations on rings preserve the property of being Noetherian.

Theorem 3.5. *If R is a Noetherian ring then so is R/I for any ideal I in R .*

Proof. Every ideal in R/I has the form J/I for an ideal J of R such that $I \subset J \subset R$. Since R is a Noetherian ring, J is finitely generated as an R -module, and those same elements reduce to a generating set of J/I as an ideal of R/I .

As an alternative proof, view R/I as an R -module in the natural way: $r(x \bmod I) = rx \bmod I$. This equals $(r \bmod I)(x \bmod I)$, so ideals in R/I are the same thing as R -submodules of R/I . We know R/I is a Noetherian *module* over R : both R and I are Noetherian R -modules, so their quotient R/I is a Noetherian R -module. Therefore R -submodules of R/I are finitely generated, which means the same thing as ideals of R/I being finitely generated. \square

In the second proof we used the fact that a Noetherian ring is a Noetherian module over itself. In fact a commutative ring is a Noetherian ring if and only if it is a Noetherian module over itself. *Make sure you understand this.*

To create more examples of Noetherian rings we can use the following very important theorem.

Theorem 3.6 (Hilbert Basis Theorem). *If R is a Noetherian ring then so is $R[X]$.*

The reason for the name ‘‘Basis Theorem’’ is historical. It’s not about bases in the modern sense, but about generating sets for modules: an ideal in $R[X]$ that is initially defined with infinitely many generators has a finite set of generators.

Proof. Let I be an ideal in $R[X]$. To prove I is finitely generated we assume it is not, so $I \neq (0)$. We will construct a sequence in I recursively. Let f_1 be any element of I with least degree. If we have f_1, \dots, f_k in I then $(f_1, \dots, f_k) \neq I$ since I is not finitely generated, so there is an f_{k+1} of least degree in $I - (f_1, \dots, f_k)$. In this way we have created a sequence of polynomials f_1, f_2, \dots in I and $\deg f_k \leq \deg f_{k+1}$ for all k . (If $\deg f_{k+1} < \deg f_k$, then our selection of f_k would be in error since f_k and f_{k+1} are both in I and outside (f_1, \dots, f_{k-1}) , and f_k is supposed to have least degree among such polynomials.)

Let a_k be the leading coefficient of f_k and consider the ideal (a_1, a_2, \dots) in R generated by all a_k ’s. Since R is a Noetherian ring, this ideal is finitely generated. Each element in this ideal is a linear combination of the a_k ’s, so there is a finite generating set for the ideal using finitely many a_k ’s. Thus $(a_1, a_2, \dots) = (a_1, a_2, \dots, a_m)$ for some m , so $a_k \in (a_1, a_2, \dots, a_m)$ for all k . We will construct a polynomial in $I - (f_1, \dots, f_m)$ with smaller degree than $\deg f_{m+1}$, which will contradict the definition of f_{m+1} , thus finishing the proof.

Since $a_{m+1} \in (a_1, a_2, \dots, a_m)$, write $a_{m+1} = \sum_{k=1}^m c_k a_k$ with $c_i \in R$. Since $\deg f_k \leq \deg f_{m+1}$ for $k \leq m$, the sum

$$\sum_{k=1}^m c_k f_k X^{\deg f_{m+1} - \deg f_k}$$

is a polynomial in (f_1, \dots, f_m) and the coefficient of $X^{\deg f_{m+1}}$ is $\sum_{k=1}^m c_k a_k = a_{m+1}$. That means

$$\sum_{k=1}^m c_k f_k X^{\deg f_{m+1} - \deg f_k} - f_{m+1}$$

is in I , is not in (f_1, \dots, f_m) , and it has smaller degree than f_{m+1} . This violates the definition of f_{m+1} , so we have a contradiction. \square

To summarize this proof in a single phrase, ‘‘use an ideal of leading coefficients’’.

The converse of Theorem 3.6 is also true: if the ring $R[X]$ is Noetherian then so is the ring R by Theorem 3.5, since $R \cong R[X]/(X)$

Corollary 3.7. *If R is a Noetherian ring then so is $R[X_1, \dots, X_n]$.*

Proof. We induct on n . The case $n = 1$ is Theorem 3.6. For $n \geq 2$, write $R[X_1, \dots, X_n]$ as $(R[X_1, \dots, X_{n-1}])[X_n]$, with $R[X_1, \dots, X_{n-1}]$ being Noetherian by the inductive hypothesis, so we are reduced to the base case. \square

Now we can build lots of Noetherian rings. The quadratic ring $\mathbf{Z}[\sqrt{d}]$ for any nonsquare integer d is Noetherian, since it's isomorphic to $\mathbf{Z}[X]/(X^2 - d)$, with $\mathbf{Z}[X]$ Noetherian by Hilbert's basis theorem and then $\mathbf{Z}[X]/(X^2 - d)$ is Noetherian by Theorem 3.5. Similarly, the rings $\mathbf{Z}[\sqrt{2}, \sqrt{3}]$ and $\mathbf{Z}[i, \sqrt[3]{2}, \sqrt[3]{10}]$ are Noetherian because they are isomorphic to $\mathbf{Z}[X, Y]/(X^2 - 2, Y^2 - 3)$ and $\mathbf{Z}[X, Y, Z]/(X^2 + 1, Y^3 - 2, Z^7 - 10)$. The ring $\mathbf{Z}[X, 1/X]$ is Noetherian since it is isomorphic to $\mathbf{Z}[X, Y]/(XY - 1)$.

For any field K and ideal I in $K[X_1, \dots, X_n]$, the ring $K[X_1, \dots, X_n]/I$ is Noetherian since K is Noetherian (in a trivial way). For instance,

$$\mathbf{R}[X, Y, Z]/(X^2 + Y^3 - Z^5, XYZ)$$

is Noetherian.

That ideals in $\mathbf{C}[X_1, \dots, X_n]$ are Noetherian was Hilbert's goal when he proved his basis theorem in 1890. Trivially $\mathbf{C}[X]$ is Noetherian, since this ring is a PID. That all ideals in $\mathbf{C}[X, Y]$ are finitely generated was known before Hilbert, using very complicated calculations. To show $\mathbf{C}[X, Y, Z]$ is Noetherian had remained an open problem for around 20 years before Hilbert's proof handled $\mathbf{C}[X_1, \dots, X_n]$ for all n in one stroke.² His proof was revolutionary for its non-constructive aspect. Even the first step, in the choice of f_1 , is not constructive. This generated much controversy about whether Hilbert's argument was really a proof! The use of non-constructive methods pioneered by Hilbert became characteristic of modern algebra for much of the 20th century. With the rise of fast computers in the late 20th century, generating sets for polynomial ideals are now computed routinely using Gröbner bases, which are essentially a multivariable replacement for the Euclidean algorithm of polynomials in one variable.

Remark 3.8. In addition to polynomials in finitely many variables, formal power series in finitely many variables are important. For any Noetherian ring R the formal power series ring $R[[X_1, \dots, X_n]]$ is Noetherian, and as in the polynomial case writing $R[[X_1, \dots, X_n]]$ as $R[[X_1, \dots, X_{n-1}]][[X_n]]$ reduces the proof to the case $n = 1$. A formal power series usually doesn't have a leading coefficient, so the proof in the polynomial case doesn't work directly for power series. What can be used with formal power series instead of a leading term is a lowest degree term, so the proof of Theorem 3.5 can be adapted to formal power series by changing highest-degree coefficients into lowest-degree coefficients. See [2, Theorem 7.11].

4. FINITELY GENERATED MODULES OVER A NOETHERIAN RING

Submodules of a finitely generated module need not be finitely generated; this in fact motivated our definition of a Noetherian module. We prove in the next theorem that when the scalar ring is Noetherian, the *a priori* weaker condition of a module being finitely generated implies the stronger condition that all of its submodules are finitely generated.

²Hilbert could not use the exact proof that we gave for his basis theorem, since he didn't have the concept of a Noetherian ring in full generality available to him.

Theorem 4.1. *If R is a Noetherian ring then an R -module is Noetherian if and only if it is finitely generated. That is, if all ideals in R are finitely generated then all submodules of an R -module M are finitely generated if and only if M is finitely generated.*

Proof. From the definition of a Noetherian module, an R -module that is Noetherian has to be finitely generated. Now suppose an R -module M is finitely generated, so M is a quotient module of some R^k . The module R^k is Noetherian by Corollary 2.5 and any quotient module of R^k is Noetherian by Theorem 2.2. Thus M is Noetherian, so all submodules of M are finitely generated. \square

Remark 4.2. The part of this proof showing finitely generated modules over Noetherian rings are Noetherian is very similar to the proof in Theorem 2.6 that finitely generated modules over a PID are Noetherian. In fact, the proofs are exactly the same: the only way the proof of Theorem 2.6 used that the ring is a PID is to be sure that the ring is a Noetherian module over itself, and that property is exactly what a Noetherian ring is.

Here is a nice application of this theorem.

Theorem 4.3. *If R is a Noetherian ring and M and N are finitely generated R -modules, then $\text{Hom}_R(M, N)$ is a finitely generated R -module.*

Proof. Since M is finitely generated, it is isomorphic to a quotient module of some R^k , say $M \cong R^k/L$. Then $\text{Hom}_R(M, N) \cong \text{Hom}_R(R^k/L, N)$. The elements of $\text{Hom}_R(R^k/L, N)$ are the R -linear maps $R^k/L \rightarrow N$. Such a map is the same thing as an R -linear map $R^k \rightarrow N$ that vanishes on L . (Make sure you understand that.) The set of all R -linear maps $R^k \rightarrow N$ vanishing on L is a submodule of $\text{Hom}_R(R^k, N)$, and the module $\text{Hom}_R(R^k/L, N)$ can be identified with it. Thus $\text{Hom}_R(M, N)$ can be embedded into $\text{Hom}_R(R^k, N)$.

Writing down an R -linear map $R^k \rightarrow N$ is the same thing as choosing a k -tuple in N and sending the standard basis of R^k there and extending by linearity, so (check!) $\text{Hom}_R(R^k, N) \cong N^k$ by $f \mapsto (f(e_1), \dots, f(e_k))$. Since N is a finitely generated R -module, N^k is also a finitely generated R -module, and $\text{Hom}_R(M, N)$ embeds as a submodule of N^k by work above. Since R is a Noetherian ring, N^k is a Noetherian R -module by Theorem 4.1, so its submodules are finitely generated. Thus $\text{Hom}_R(M, N)$ is finitely generated. \square

If we drop the condition that R is a Noetherian ring, it can be false that $\text{Hom}_R(M, N)$ is finitely generated when M and N are.

Example 4.4. For any ideal I in R we have $\text{Hom}_R(R/I, R) \cong \{r \in R : Ir = 0\}$ (an R -linear map out of R/I is determined by where $\bar{1}$ goes), so we will give an example of an R and I where $\{r \in R : Ir = 0\}$ is not finitely generated. Both R and R/I are finitely generated R -modules, since each is generated by 1.

Let K be a field and $R = K[X_1, X_2, \dots]/(\dots, X_i X_j, \dots)$. Let I be the ideal of polynomial cosets in R with constant term 0. (The constant term of a coset is well-defined $X_i X_j$ has constant term 0.) Then $If = 0$ if and only if f has constant term 0, so $\{f \in R : If = 0\} = I$. That the ideal I is not finitely generated is very similar to the proof that (X_1, X_2, \dots) in $K[X_1, X_2, \dots]$ is not finitely generated.

Corollary 4.5. *If R is a Noetherian ring and M and N are Noetherian R -modules, then $\text{Hom}_R(M, N)$ is a Noetherian R -module.*

Proof. Combine Theorems 4.1 and 4.3. \square

Corollary 4.6. *If R is a Noetherian ring and M is a finitely generated R -module then its dual module M^\vee is finitely generated.*

Proof. Apply Theorem 4.3 with $N = R$. □

Without the Noetherian condition on R , Corollary 4.6 can break down. Example 4.4 uses a dual module, so a finitely generated module might not have a finitely generated dual module when R is a non-Noetherian ring.

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