# COMPUTING THE NORM OF A MATRIX

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## 1. Norms on Vector Spaces

Let $V$ be a vector space over $\mathbf{R}$. A *norm* on $V$ is a function $||\cdot||\colon V \to \mathbf{R}$ satisfying three properties:

1) $||v|| \geq 0$, with equality if and only if $v = 0$,
2) $||v + w|| \leq ||v|| + ||w||$ for $v, w \in V$,
3) $||\alpha v|| = |\alpha| ||v||$ for $\alpha \in \mathbf{R}$, $v \in V$.

The same definition applies to a complex vector space. From a norm we get a metric on $V$ by $d(v, w) = ||v - w||$.

The standard norm on $\mathbf{R}^n$ is

$$\left|\left|\sum_{i=1}^{n} a_i e_i\right|\right| = \sqrt{\sum_{i=1}^{n} a_i^2}.$$

This gives rise to the Euclidean metric on $\mathbf{R}^n$. Another norm on $\mathbf{R}^n$ is the sup-norm:

$$\left|\left|\sum_{i=1}^{n} a_i e_i\right|\right|_{\sup} = \max_i |a_i|.$$

This gives rise to the sup-metric on $\mathbf{R}^n$: $d(\sum a_i e_i, \sum b_i e_i) = \max |a_i - b_i|$.

On $\mathbf{C}^n$ the standard norm is

$$\left|\left|\sum_{i=1}^{n} a_i e_i\right|\right| = \sqrt{\sum_{i=1}^{n} |a_i|^2},$$

and the sup-norm is defined as on $\mathbf{R}^n$.

A common way of placing a norm on a real vector space $V$ is via an *inner product*, which is a pairing $(\cdot,\cdot)\colon V \times V \to \mathbf{R}$ that is

1) bilinear,
2) symmetric: $(v, w) = (w, v)$,

and

3) positive-definite: $(v, v) \geq 0$, with equality if and only if $v = 0$.

The standard inner product on $\mathbf{R}^n$ is

$$\left(\sum_{i=1}^{n} a_i e_i, \sum_{i=1}^{n} b_i e_i\right) = \sum_{i=1}^{n} a_i b_i.$$

For an inner product $(\cdot,\cdot)$ on $V$, a norm can be defined by the formula

$$||v|| = \sqrt{(v, v)}.$$

That this actually is a norm is a consequence of the Cauchy-Schwarz inequality, $|(v, w)| \leq \sqrt{(v, v)(w, w)} = ||v|| ||w||$, whose proof can be found in most linear algebra books. In particular, using the standard inner product on $\mathbf{R}^n$ we get the classic form of this inequality, as proven by Cauchy:

$$\left|\sum_{i=1}^{n} a_i b_i\right| \leq \sqrt{\sum_{i=1}^{n} a_i^2 \cdot \sum_{i=1}^{n} b_i^2}.$$

However, the Cauchy-Schwarz inequality is true for any inner product on a real-vector space, not just the standard inner product on $\mathbf{R}^n$.

The norm on $\mathbf{R}^n$ that comes from the standard inner product is the standard norm. On the other hand, the sup-norm on $\mathbf{R}^n$ does *not* arise from an inner product, i.e. there is no inner product whose associated norm is the sup-norm.

Although the sup-norm and the standard norm on $\mathbf{R}^n$ are not equal, they are each bounded by a constant multiple of the other one:

$$\max_i |a_i| \leq \sqrt{\sum_{i=1}^{n} a_i^2} \leq \sqrt{n} \max_i |a_i|,$$

i.e. $||v||_{\text{sup}} \leq ||v|| \leq \sqrt{n}||v||_{\text{sup}}$. Therefore the metrics these two norms give rise to determine the same notions of convergence: a sequence in $\mathbf{R}^n$ which is convergent with respect to one of the metrics is also convergent with respect to the other metric.

The standard inner product on $\mathbf{R}^n$ is closely tied to transposition of $n$ by $n$ matrices. For $A = (a_{ij}) \in \mathrm{M}_n(\mathbf{R})$, let $A^\top = (a_{ji})$ be its transpose. Then for any $v, w \in \mathbf{R}^n$,

$$(Av, w) = (v, A^\top w),$$

where $(\cdot,\cdot)$ is the standard inner product.

Let's briefly indicate what the analogue of these ideas is for complex vector spaces. An *inner product* on a complex vector space $V$ is a pairing $(\cdot,\cdot)\colon V \times V \to \mathbf{C}$ which is

1) linear on the left and conjugate-linear on the right: $(v, \alpha w) = \overline{\alpha}(v, w)$ for $\alpha \in \mathbf{C}$,
2) skew-symmetric: $(v, w) = \overline{(w, v)}$,
3) positive-definite: $(v, v) \geq 0$, with equality if and only if $v = 0$.

Physicists usually have inner products on complex vector spaces being linear on the right and conjugate linear on the left. It's just a difference in notation.

The standard inner product on $\mathbf{C}^n$ is

$$\left(\sum_{i=1}^{n} a_i e_i, \sum_{i=1}^{n} b_i e_i\right) = \sum_{i=1}^{n} a_i \overline{b_i}.$$

An inner product on a complex vector space also satisfies the Cauchy-Schwarz inequality, so can be used to define a norm just as in the case of inner products on real vector spaces.

The above inner product on $\mathbf{C}^n$ is closely tied to conjugate-transposition of $n$ by $n$ complex matrices. For $A = (a_{ij}) \in \mathrm{M}_n(\mathbf{C})$, let $A^* = (\overline{a_{ji}})$ be its conjugate-transpose. Then for any $v, w \in \mathbf{C}^n$,

$$(Av, w) = (v, A^* w).$$

Although we will be focusing on norms on finite-dimensional spaces, the extension to infinite-dimensional spaces is quite important.

*From now on, the norm and inner product on $\mathbf{R}^n$ and $\mathbf{C}^n$ are the standard ones.*

## 2. Defining Norms on Matrices

How should we define the size of an $n$ by $n$ real matrix $A = (a_{ij})$? The simplest idea is to view $\mathrm{M}_n(\mathbf{R})$ as $\mathbf{R}^{n^2}$ and just use the sup-norm:

$$||(a_{ij})||_{\sup} = \max_{i,j} |a_{ij}|.$$

It turns out that this is not the best norm to put on matrices. Neither is the standard norm $||(a_{ij})|| = \sqrt{\sum a_{ij}^2}$. However, before indicating the right norm for matrices, let's use the idea of the sup-norm to prove that matrices define continuous maps $\mathbf{R}^n \to \mathbf{R}^n$. For $v \in \mathbf{R}^n$ with $j$th coordinate $v_j$,

$$\begin{aligned} ||Av|| &\leq \sqrt{n}||Av||_{\sup} \\ &= \sqrt{n} \max_i \left| \sum_{j=1}^n a_{ij} v_j \right| \\ &\leq \sqrt{n} \max_i \sum_{j=1}^n |a_{ij}||v_j| \\ &\leq n\sqrt{n} \max_{i,j} |a_{ij}| \cdot ||v||_{\sup} \\ &\leq n\sqrt{n} \max_{i,j} |a_{ij}| \cdot ||v||. \end{aligned}$$

Let $C = n\sqrt{n} \max |a_{ij}|$. This is a constant depending on the dimension $n$ of the space and the matrix $A$, but not on $v$. By linearity, $||Av - Aw|| \leq C||v - w||$ for all $v, w \in \mathbf{R}^n$, so if $v \to w$ then $Av \to Aw$. Thus $A\colon \mathbf{R}^n \to \mathbf{R}^n$ is continuous.

It turns out that the right way to define a norm on $\mathrm{M}_n(\mathbf{R})$ is to first choose a norm on $\mathbf{R}^n$ and then define a norm on $\mathrm{M}_n(\mathbf{R})$ based on this choice. To keep matters concrete, we will use the standard norm on $\mathbf{R}^n$.

Let's write down the above calculation as a lemma.

**Lemma 2.1.** *For $A \in \mathrm{M}_n(\mathbf{R})$, there is a $C \geq 0$ such that $||Av|| \leq C||v||$ for all $v \in \mathbf{R}^n$.*

Of course the constant $C$ we wrote down might not be optimal. Perhaps there is a smaller constant $C' < C$ such that $||Av|| \leq C'||v||$ for all $v \in \mathbf{R}^n$. Roughly speaking, from the standard norm $|| \cdot ||$ on $\mathbf{R}^n$ we get a norm on $\mathrm{M}_n(\mathbf{R})$ by assigning to $A \in \mathrm{M}_n(\mathbf{R})$ the least $C \geq 0$ such that $||Av|| \leq C||v||$ for all $v \in \mathbf{R}^n$.

**Theorem 2.2.** *For any $A \in \mathrm{M}_n(\mathbf{R})$, there is a unique real number $b$ such that (i) $||Av|| \leq b||v||$ for all $v \in \mathbf{R}^n$ and (ii) if $||Av|| \leq C||v||$ for all $v \in \mathbf{R}^n$, then $b \leq C$.*

*Proof.* By scaling for nonzero vectors, the inequality $||Av|| \leq C||v||$ is true for all $v \in \mathbf{R}^n$ if and only if the inequality $||Av|| \leq C$ is true for all $v \in \mathbf{R}^n$ with $||v|| = 1$. So this theorem is saying that the set $\{||Av|| : ||v|| = 1\}$ attains a (finite) maximum value. This is what we will prove.

Let $b$ be the least upper bound of this set, so certainly $||Av|| \leq b$ for all $v$ with $||v|| = 1$. We must show $b = ||Ax||$ for some $x \in \mathbf{R}^n$ with $||x|| = 1$. Choose a sequence $v_n$ such that $||v_n|| = 1$ and $||Av_n|| \to b$. Since the unit ball in $\mathbf{R}^n$ is compact, $\{v_n\}$ has a limit point, say $x$. Then by continuity of $A$, $Ax$ is a limit point of the sequence $\{Av_n\}$. Since $||Av_n|| \to b$, it follows that $||Ax|| = b$. $\square$

**Definition 2.3.** For $A \in \mathrm{M}_n(\mathbf{R})$, $||A||$ is the smallest real number satisfying the inequality $||Av|| \leq ||A||||v||$ for all $v \in \mathbf{R}^n$. This is called the *operator norm* of $A$.

The next theorem shows the operator norm is a vector space norm on $\mathrm{M}_n(\mathbf{R})$ and has a host of other nice properties.

**Theorem 2.4.** *For $A, B \in \mathrm{M}_n(\mathbf{R})$ and $v, w \in \mathbf{R}^n$,*
*i) $||A|| \geq 0$, with equality if and only if $A = 0$.*
*ii) $||A + B|| \leq ||A|| + ||B||$.*
*iii) $||\alpha A|| = |\alpha|||A||$ for $\alpha \in \mathbf{R}$.*
*iv) $||AB|| \leq ||A||||B||$. It is typically false that $||AB|| = ||A||||B||$.*
*v) $||A|| = ||A^\top||$.*
*vi) $||AA^\top|| = ||A^\top A|| = ||A||^2$. Thus $||A|| = \sqrt{||AA^\top||} = \sqrt{||A^\top A||}$.*
*vii) $|(Av, w)| \leq ||A||||v||||w||$.*
*viii) $||A||_{\sup} \leq ||A|| \leq n\sqrt{n}||A||_{\sup}$, and $\mathrm{M}_n(\mathbf{R})$ is complete with respect to the operator norm.*

*Proof.* i) If $||A|| = 0$ then for all $v \in \mathbf{R}^n$, $||Av|| \leq 0||v|| = 0$, so $Av = 0$, so $A = 0$. The converse is trivial.

ii) For $v \in \mathbf{R}^n$,

$$\begin{aligned} ||(A+B)v|| &\leq ||Av + Bv|| \\ &\leq ||Av|| + ||Bv|| \\ &\leq ||A||||v|| + ||B||||v|| \\ &= (||A|| + ||B||)||v||. \end{aligned}$$

Therefore $||A + B|| \leq ||A|| + ||B||$.

iii) Left to the reader.

iv) For $v \in \mathbf{R}^n$, $||(AB)v|| = ||A(Bv)|| \leq ||A||||Bv|| \leq ||A||||B||||v||$, so the basic property of the operator norm implies $||AB|| \leq ||A||||B||$.

To show that generally $||AB|| \neq ||A||||B||$, note that if $||AB|| = ||A||||B||$ for all $A, B$, then for $A, B \neq 0$ we'd have $||AB|| \neq 0$, so $AB \neq 0$. That is, the product of any two nonzero matrices is nonzero. This is false when $n > 1$, since there are many nonzero matrices whose square is zero.

v) $||Av||^2 = |(Av, Av)| = |(v, A^\top Av)| \leq ||v||||A^\top Av||$ by Cauchy-Schwarz. This last expression is $\leq ||A^\top A||||v||^2$, so

$$||Av|| \leq \sqrt{||A^\top A||}||v||.$$

Therefore $||A|| \leq \sqrt{||A^\top A||}$, so

$$||A||^2 \leq ||A^\top A|| \leq ||A^\top||||A||.$$

Dividing by $||A||$ when $A \neq 0$, we get $||A|| \leq ||A^\top||$. This is also obvious for $A = 0$. Now replacing $A$ by $A^\top$, we get

$$||A^\top|| \leq ||(A^\top)^\top|| = ||A||.$$

So $||A|| = ||A^\top||$.

vi) Going back to part v,

$$||A||^2 \leq ||A^\top A|| \leq ||A^\top||||A|| = ||A||^2.$$

Therefore $||A||^2 = ||A^\top A||$. Using $A^\top$ in place of $A$ we get the other inequality, since $||A^\top|| = ||A||$.

vii) Use Cauchy-Schwarz.

viii) Set $v = e_j$ and $w = e_i$ in part vii:

$$|a_{ij}| \leq ||A||.$$

Therefore $||A||_{\text{sup}} \leq ||A||$. The other inequality follows from the calculation preceding Lemma 2.1.

That $\mathrm{M}_n(\mathbf{R})$ is complete with respect to the operator norm follows from completeness with respect to the sup-norm and the fact that these two norms are bounded by constant multiples of each other. □

So we have a norm on $\mathrm{M}_n(\mathbf{R})$ which interacts nicely with the ring structure and the standard inner product on $\mathbf{R}^n$ (parts iv through vii of Theorem 2.4). However, unlike the standard norm on $\mathbf{R}^n$, the operator norm on $\mathrm{M}_n(\mathbf{R})$ is impossible to calculate from the definition in all but the simplest cases. For instance, it is clear that $||I_n|| = 1$, so $||\alpha I_n|| = |\alpha|$ for any real number $\alpha$. But what is

$$\left|\left|\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\right|\right|?$$

By the last part of Theorem 2.4, this norm is bounded above by $8\sqrt{2}$. In the next section we give a computational formula for the operator norm on $\mathrm{M}_n(\mathbf{R})$ which will allow us to compute the norm of this matrix very easily.

**Exercise.** Define an operator norm on $\mathrm{M}_n(\mathbf{C})$ and establish an analogue of Theorem 2.4. Is it true that a matrix in $\mathrm{M}_n(\mathbf{C})$ generally has the same operator norm as its transpose? For a real $n$ by $n$ matrix, show its operator norm as an element of $\mathrm{M}_n(\mathbf{R})$ equals its operator norm as an element of $\mathrm{M}_n(\mathbf{C})$.

## 3. A Computational Formula for a Matrix Norm

The key idea to compute the operator norm of $A \in \mathrm{M}_n(\mathbf{R})$ is that this is an explicit function of the operator norm of $AA^\top$, by Theorem 2.4 vi, and it turns out that the operator norm of a matrix of the form $AA^\top$ is its largest eigenvalue, so our computation reduces to locating the roots of a characteristic polynomial.

What makes $AA^\top$ special is that it equals its own transpose:

$$(AA^\top)^\top = (A^\top)^\top A^\top = AA^\top.$$

The following theorem, due to Peano, gives a method to compute operator norms.

**Theorem 3.1** (Peano). 1) *If $A \in \mathrm{M}_n(\mathbf{R})$ satisfies $A^\top = A$, then all the eigenvalues of $A$ are real and*

$$||A|| = \max_{\text{eigenvalues } \lambda \text{ of } A} |\lambda|.$$

2) *For any $A \in \mathrm{M}_n(\mathbf{R})$, the eigenvalues of $AA^\top$ are all nonnegative, so $||A|| = \sqrt{||AA^\top||}$ is the square root of the largest eigenvalue of $AA^\top$.*

*Proof:* 1) To prove all the eigenvalues of $A$ are real, i.e. all the roots of the characteristic polynomial $\det(XI_n - A)$ are real, let $A$ act on $\mathbf{C}^n$ in the obvious way. Using the standard inner product on $\mathbf{C}^n$, we have for all $v \in \mathbf{C}^n$ that

$$(Av, v) = (v, \overline{A}^\top v) = (v, A^\top v) = (v, Av).$$

For an eigenvalue $\lambda \in \mathbf{C}$ of $A$, let $v \in \mathbf{C}^n$, $v \neq 0$, be a corresponding eigenvector. Then

$$(Av, v) = (\lambda v, v) = \lambda(v, v), \quad (v, Av) = (v, \lambda v) = \overline{\lambda}(v, v),$$

so $\lambda = \overline{\lambda}$ since $(v, v) \neq 0$. Thus $\lambda$ is real.

Now we need to show

i) For an eigenvalue $\lambda$ of $A$, $|\lambda| \leq ||A||$,

ii) $||A||$ or $-||A||$ is an eigenvalue of $A$.

The first property is a general fact about matrices, so we'll do it in $\mathrm{M}_n(\mathbf{C})$ rather than in $\mathrm{M}_n(\mathbf{R})$ since eigenvalues of general matrices are complex, not real. The second one will use $A = A^\top$ in an essential way.

For i, assume $|\lambda| > ||A||$. We'll assume the reader has done the exercise at the end of the previous section, so we can regard $||A||$ as the operator norm of $A$ viewed as an element of $\mathrm{M}_n(\mathbf{C})$. Then

$$\begin{aligned} \left|\left|\left(\frac{1}{\lambda}A\right)^n\right|\right| &= \frac{1}{|\lambda|^n}||A^n|| \\ &\leq \frac{1}{|\lambda|^n}||A||^n \\ &< 1. \end{aligned}$$

Therefore the geometric series $\sum_{n \geq 0}(\frac{1}{\lambda})^n A^n$ converges in $\mathrm{M}_n(\mathbf{C})$, to a multiplicative inverse for $I_n - \frac{1}{\lambda}A$. (Details are left to the reader. Show the partial sums of the series are a Cauchy sequence in $\mathrm{M}_n(\mathbf{C})$ and then use completeness of $\mathrm{M}_n(\mathbf{C})$ with respect to the operator norm.) Then $\lambda I_n - A$ has a multiplicative inverse, so it must have nonzero determinant. In particular, $\lambda$ is not an eigenvalue of $A$. So if $\lambda$ is an eigenvalue of $A$, we must have $|\lambda| \leq ||A||$.

For ii, we begin by noting that for any vector $v \in \mathbf{R}^n$ of size 1, $|(Av, v)| \leq ||A||$. We want to show that there is a sequence $\{v_n\} \subset \mathbf{R}^n$ such that $||v_n|| = 1$ and $|(Av_n, v_n)| \to ||A||$. This will use the assumption that $A = A^\top$.

The set $\{|(Av, v)| : ||v|| = 1\}$ is bounded above by $||A||$. Let $c$ be the least upper bound of this set, so $c \leq ||A||$ and $|(Av, v)| \leq c$ for all $v \in \mathbf{R}^n$ with $||v|| = 1$. We will conclude from this that $||Av|| \leq c||v||$ for all $v \in \mathbf{R}^n$, so $||A|| \leq c$. Thus $c = ||A||$, so $||A||$ is a limit point, as desired.

Since $|(Av, v)| \leq c$ for vectors $v$ of size 1, scaling implies $|(Av, v)| \leq c||v||^2$ for all $v \in \mathbf{R}^n$. We now use the Polarization Identity, valid for any inner product and any linear map:

$$2((Av, w) + (Aw, v)) = (A(v + w), v + w) + (A(v - w), v - w).$$

Just expand the right hand side by additivity to get the left hand side, thus verifying the identity.

When $A = A^\top$, the left hand side becomes $4(Av, w)$. Therefore

$$\begin{aligned} 4|(Av,w)| &\leq c||v+w||^2 + c||v-w||^2 \\ &= c((v+w,v+w) + (v-w,v-w)) \\ &= 2c((v,v)+(w,w)) \\ &= 2c(||v||^2 + ||w||^2). \end{aligned}$$

For $||v|| = ||w|| = 1$, this implies $4|(Av,w)| \leq 4c$, so $|(Av,w)| \leq c$ if $||v|| = ||w|| = 1$. Scaling, this implies for any $v, w \in \mathbf{R}^n$ that $|(Av,w)| \leq c||v||||w||$. Now choose $w = Av$. We get

$$||Av||^2 = |(Av, Av)| \leq c||v||||Av||.$$

Therefore $||Av|| \leq c||v||$ if $Av \neq 0$. It is also obvious if $Av = 0$. Then by the basic property of the operator norm, $||A|| \leq c$. Thus $||A|| = \lim_{n\to\infty} |(Av_n, v_n)|$ for some sequence of unit vectors $v_n$ in $\mathbf{R}^n$.

Since the unit ball in $\mathbf{R}^n$ is compact, we can pass to a subsequence and therefore assume that the sequence $\{v_n\}$ converges, say to $x$. Necessarily $||x|| = 1$. Since the numbers $(Av_n, v_n)$ are real, we can pass to a subsequence and assume that $(Av_n, v_n) \to \alpha$, where $\alpha = ||A||$ or $-||A||$. From

$$\begin{aligned} 0 &\leq (Av_n - \alpha v_n, Av_n - \alpha v_n) \\ &= (Av_n, Av_n) - 2\alpha(Av_n, v_n) + \alpha^2(v_n, v_n) \\ &= ||Av_n||^2 - 2\alpha(Av_n, v_n) + \alpha^2 \\ &\leq ||A||^2 - 2\alpha(Av_n, v_n) + \alpha^2 \\ &= 2(\alpha^2 - \alpha(Av_n, v_n)), \end{aligned}$$

we conclude that as $n \to \infty$, $Av_n - \alpha v_n \to 0$. We already have that $v_n \to x$, so $Av_n \to Ax$ by continuity of $A$. Therefore $\alpha v_n \to Ax$. Since $\alpha v_n \to \alpha x$ as well, we must have $Ax = \alpha x$. Since $||x|| = 1$, $x \neq 0$, so $\alpha = \pm||A||$ is an eigenvalue of $A$.

2) We just show the eigenvalues of $AA^\top$ are all nonnegative. The rest follows from the other work we have done. If $\lambda$ is an eigenvalue, let $v \in \mathbf{C}^n$ be an eigenvector. Then using the standard inner product on $\mathbf{C}^n$,

$$0 \leq (A^\top v, A^\top v) = (AA^\top v, v) = (\lambda v, v) = \lambda(v,v).$$

Since $(v,v) \geq 0$, it follows that $\lambda \geq 0$.

Of course, since we already know the eigenvalues of $AA^\top$ are real, we could have taken the eigenvector $v$ to be in $\mathbf{R}^n$ and used the standard inner product on $\mathbf{R}^n$ instead. But that would not remove the need to consider $\mathbf{C}^n$ within the context of this problem, since complex $n$-space was used to prove the eigenvalues of a symmetric matrix like $AA^\top$ are all real in part 1.

□

**Exercise.** Generalize Theorem 3.1 to give a computational formula for the operator norm of matrices in $\mathrm{M}_n(\mathbf{C})$.

**Example.** Let's compute the operator norm of the 2 by 2 matrix

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

Since

$$AA^\top = \begin{pmatrix} 5 & 11 \\ 11 & 25 \end{pmatrix},$$

the characteristic polynomial of $AA^\top$ is $X^2 - 30X + 4$, whose largest eigenvalue is $15 + \sqrt{221}$. Therefore the operator norm of $A$ is $\sqrt{15 + \sqrt{221}}$. That is, for every $(x, y) \in \mathbf{R}^2$,

$$||(5x + 11y, 11x + 25y)|| \leq \sqrt{15 + \sqrt{221}}||(x, y)||,$$

and $\sqrt{15 + \sqrt{221}}$ is the smallest number with this property. Try computing this operator norm from the definition!

To summarize, Peano's theorem reduces the computation of an operator norm to the computation of the largest root of a (monic) polynomial whose roots are known to be nonnegative. So to program a computer to calculate an operator norm, we need an upper bound on the size of the roots of a polynomial, and such bounds exist in terms of the size of the coefficients. It is left to the reader to find such bounds in the literature or produce them anew.