THE SYLOW THEOREMS

1. Introduction

The converse of Lagrange's theorem is false: if G is a finite group and $d \mid |G|$, then there may not be a subgroup of G with order d. The simplest example of this is the group A_4, of order 12, which has no subgroup of order 6. The Norwegian mathematician Peter Ludwig Sylow [1] discovered that a converse result *is* true when d is a prime power: if p is a prime number and $p^k \mid |G|$ then G must contain a subgroup of order p^k. Sylow also discovered important relations among the subgroups with order the *largest* power of p dividing $|G|$, such as the fact that all subgroups of that order are conjugate to each other.

For example, a group of order $100 = 2^2 \cdot 5^2$ must contain subgroups of order 1, 2, 4, 5, and 25, the subgroups of order 4 are conjugate to each other, and the subgroups of order 25 are conjugate to each other. It is not necessarily the case that the subgroups of order 2 are conjugate or that the subgroups of order 5 are conjugate.

Definition 1.1. Let G be a finite group and p be a prime. Any subgroup of G whose order is the highest power of p dividing $|G|$ is called a *p-Sylow subgroup* of G. A p-Sylow subgroup for some p is called a *Sylow subgroup*.

In a group of order 100, a 2-Sylow subgroup has order 4, a 5-Sylow subgroup has order 25, and a p-Sylow subgroup is trivial if $p \neq 2$ or 5.

In a group of order 12, a 2-Sylow subgroup has order 4, a 3-Sylow subgroup has order 3, and a p-Sylow subgroup is trivial if $p > 3$. Let's look at a few examples of Sylow subgroups in groups of order 12.

Example 1.2. In $\mathbf{Z}/(12)$, the only 2-Sylow subgroup is $\{0, 3, 6, 9\} = \langle 3 \rangle$ and the only 3-Sylow subgroup is $\{0, 4, 8\} = \langle 4 \rangle$.

Example 1.3. In A_4 there is one subgroup of order 4, so the only 2-Sylow subgroup is

$$\{(1),\ (12)(34),\ (13)(24),\ (14)(23)\} = \langle (12)(34), (14)(23) \rangle.$$

There are four 3-Sylow subgroups:

$$\{(1),\ (123),\ (132)\} = \langle (123) \rangle, \quad \{(1),\ (124),\ (142)\} = \langle (124) \rangle,$$
$$\{(1),\ (134),\ (143)\} = \langle (134) \rangle, \quad \{(1),\ (234),\ (243)\} = \langle (234) \rangle.$$

Example 1.4. In D_6 there are three 2-Sylow subgroups:

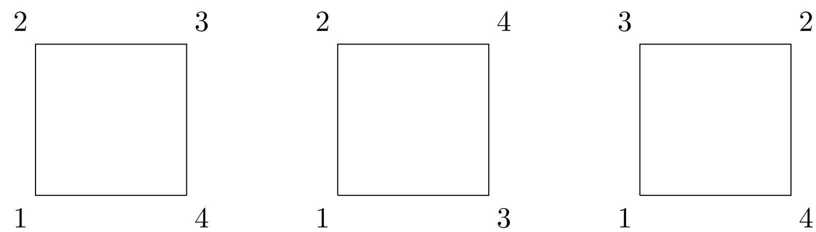
$$\{1,\ r^3,\ s,\ r^3 s\} = \langle r^3, s \rangle, \quad \{1,\ r^3,\ rs,\ r^4 s\} = \langle r^3, rs \rangle, \quad \{1,\ r^3,\ r^2 s,\ r^5 s\} = \langle r^3, r^2 s \rangle.$$

The only 3-Sylow subgroup of D_6 is $\{1, r^2, r^4\} = \langle r^2 \rangle$.

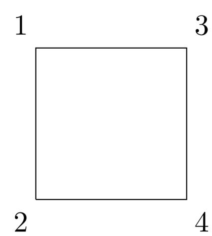
In a group of order 24, a 2-Sylow subgroup has order 8 and a 3-Sylow subgroup has order 3. Let's look at two examples.

Example 1.5. In S_4, the 3-Sylow subgroups are the 3-Sylow subgroups of A_4 (an element of 3-power order in S_4 must be a 3-cycle, and they all lie in A_4). We determined the 3-Sylow subgroups of A_4 in Example 1.3; there are four of them.

There are three 2-Sylow subgroups of S_4, and they are interesting to work out since they can be understood as *copies of* D_4 *inside* S_4. The number of ways to label the four vertices of a square as 1, 2, 3, and 4 is $4! = 24$, but up to rotations and reflections of the square there are really just three different ways of carrying out the labeling, as follows.



Any other labeling of the square is a rotated or reflected version of one of these three squares. For example, the square below is obtained from the middle square above by reflecting across a horizontal line through the middle of the square.



When D_4 acts on a square with labeled vertices, each motion of D_4 creates a permutation of the four vertices, and this permutation is an element of S_4. For example, a 90 degree rotation of the square is a 4-cycle on the vertices. In this way we obtain a copy of D_4 inside S_4. The three essentially different labelings of the vertices of the square above embed D_4 into S_4 as three different subgroups of order 8:

$$\{1,\ (1234),\ (1432),\ (12)(34),\ (13)(24),\ (14)(23),\ (13),\ (24)\} = \langle (1234), (13)\rangle,$$

$$\{1,\ (1243),\ (1342),\ (12)(34),\ (13)(24),\ (14)(23),\ (14),\ (23)\} = \langle (1243), (14)\rangle,$$

$$\{1,\ (1324),\ (1423),\ (12)(34),\ (13)(24),\ (14)(23),\ (12),\ (34)\} = \langle (1324), (12)\rangle.$$

These are the 2-Sylow subgroups of S_4.

Example 1.6. The group $\mathrm{SL}_2(\mathbf{Z}/(3))$ has order 24. An explicit tabulation of the elements of this group reveals that there are only 8 elements in the group with 2-power order:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix},$$

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}.$$

These form the only 2-Sylow subgroup, which is isomorphic to Q_8 by labeling the matrices in the first row as $1, i, j, k$ and the matrices in the second row as $-1, -i, -j, -k$.

There are four 3-Sylow subgroups: $\langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\rangle$, $\langle \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)\rangle$, $\langle \left(\begin{smallmatrix} 0 & 1 \\ 2 & 2 \end{smallmatrix}\right)\rangle$, and $\langle \left(\begin{smallmatrix} 0 & 2 \\ 1 & 2 \end{smallmatrix}\right)\rangle$.

Here are the Sylow theorems. They are often given in three parts. The result we call Sylow III* is not always stated explicitly as part of the Sylow theorems.

Theorem 1.7 (Sylow I). *A finite group G has a p-Sylow subgroup for every prime p and any p-subgroup of G lies in a p-Sylow subgroup of G.*

Theorem 1.8 (Sylow II). *For each prime p, the p-Sylow subgroups of G are conjugate.*

Theorem 1.9 (Sylow III). *For each prime p, let n_p be the number of p-Sylow subgroups of G. Write $|G| = p^k m$, where p doesn't divide m. Then*

$$n_p \equiv 1 \bmod p \quad \text{and} \quad n_p \mid m.$$

Theorem 1.10 (Sylow III*). *For each prime p, let n_p be the number of p-Sylow subgroups of G. Then $n_p = [G : \mathrm{N}(P)]$, where P is any p-Sylow subgroup and $\mathrm{N}(P)$ is its normalizer.*

Sylow II says for two p-Sylow subgroups H and K of G that there is some $g \in G$ such that $gHg^{-1} = K$. This is illustrated in the table below.

Example	Group	Size	p	H	K	g
1.3	A_4	12	3	$\langle(123)\rangle$	$\langle(124)\rangle$	(243)
1.4	D_6	12	2	$\langle r^3, s\rangle$	$\langle r^3, rs\rangle$	r^2
1.5	S_4	24	2	$\langle(1234), (13)\rangle$	$\langle(1243), (14)\rangle$	(34)
1.6	$\mathrm{SL}_2(\mathbf{Z}/(3))$	24	3	$\langle\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\rangle$	$\langle\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)\rangle$	$\left(\begin{smallmatrix}0&1\\2&1\end{smallmatrix}\right)$

When trying to conjugate one cyclic subgroup to another cyclic subgroup, be careful: not all generators of the two groups have to be conjugate. For example, in A_4 the subgroups $\langle(123)\rangle = \{(1),\ (123),\ (132)\}$ and $\langle(124)\rangle = \{(1),\ (124),\ (142)\}$ are conjugate, but the conjugacy class of (123) in A_4 is $\{(123), (142), (134), (243)\}$, so there's no way to conjugate (123) to (124) by an element of A_4; we must conjugate (123) to (142). The 3-cycles (123) and (124) are conjugate in S_4, but not in A_4. Similarly, $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$ are conjugate in $\mathrm{GL}_2(\mathbf{Z}/(3))$ but not in $\mathrm{SL}_2(\mathbf{Z}/(3))$, so when Sylow II says the subgroups $\langle\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\rangle$ and $\langle\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)\rangle$ are conjugate in $\mathrm{SL}_2(\mathbf{Z}/(3))$ a conjugating matrix must send $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ to $\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)^2 = \left(\begin{smallmatrix}1&0\\2&1\end{smallmatrix}\right)$.

Let's see what Sylow III tells us about the number of 2-Sylow and 3-Sylow subgroups of a group of order 12. For $p = 2$ and $p = 3$ in Sylow III, the divisibility conditions are $n_2 \mid 3$ and $n_3 \mid 4$ and the congruence conditions are $n_2 \equiv 1 \bmod 2$ and $n_3 \equiv 1 \bmod 3$. The divisibility conditions imply n_2 is 1 or 3 and n_3 is 1, 2, or 4. The congruence $n_2 \equiv 1 \bmod 2$ tells us nothing new (1 and 3 are both odd), but the congruence $n_3 \equiv 1 \bmod 3$ rules out the option $n_3 = 2$. Therefore n_2 is 1 or 3 and n_3 is 1 or 4 when $|G| = 12$. If $|G| = 24$ we again find n_2 is 1 or 3 while n_3 is 1 or 4. (For instance, from $n_3 \mid 8$ and $n_3 \equiv 1 \bmod 3$ the only choices are $n_3 = 1$ and $n_3 = 4$.) Therefore as soon as we find more than one 2-Sylow subgroup there must be three of them, and as soon as we find more than one 3-Sylow subgroup there must be four of them. The table below shows the values of n_2 and n_3 in the examples above.

Group	Size	n_2	n_3
$\mathbf{Z}/(12)$	12	1	1
A_4	12	1	4
D_6	12	3	1
S_4	24	3	4
$\mathrm{SL}_2(\mathbf{Z}/(3))$	24	1	4

2. Proof of the Sylow Theorems

Our proof of the Sylow theorems will use group actions. The table below is a summary. For each theorem the table lists a group, a set it acts on, and the action. We write $\mathrm{Syl}_p(G)$ for the set of p-Sylow subgroups of G, so $n_p = |\operatorname{Syl}_p(G)|$.

Theorem	Group	Set	Action
Sylow I	p-subgroup H	G/H	left mult.
Sylow II	p-Sylow subgroup Q	G/P	left mult.
Sylow III ($n_p \equiv 1 \bmod p$)	$P \in \mathrm{Syl}_p(G)$	$\mathrm{Syl}_p(G)$	conjugation
Sylow III ($n_p \mid m$)	G	$\mathrm{Syl}_p(G)$	conjugation
Sylow III*	G	$\mathrm{Syl}_p(G)$	conjugation

The two conclusions of Sylow III are listed separately in the table since they are proved using different group actions.

Our proofs will usually involve the action of a p-group on a set and use the fixed-point congruence for such actions: $|X| \equiv |\operatorname{Fix}_\Gamma(X)| \bmod p$, where X is a finite set being acted on by a finite p-group Γ.

Proof of Sylow I: Let p^k be the highest power of p in $|G|$. The result is obvious if $k = 0$, since the trivial subgroup is a p-Sylow subgroup, so we can take $k \geq 1$, hence $p \mid |G|$.

Our strategy for proving Sylow I is to **prove a stronger result**: *there is a subgroup of order p^i for $0 \leq i \leq k$.* More specifically, if $|H| = p^i$ and $i < k$, we will show there is a p-subgroup $H' \supset H$ with $[H' : H] = p$, so $|H'| = p^{i+1}$. Then, starting with H as the trivial subgroup, we can repeat this process with H' in place of H to create a rising tower of subgroups

$$\{e\} = H_0 \subset H_1 \subset H_2 \subset \cdots$$

where $|H_i| = p^i$, and after k steps we reach H_k, which is a p-Sylow subgroup of G.

Consider the left multiplication action of H on the left cosets G/H (this need not be a group). This is an action of a finite p-group H on the set G/H, so by the fixed-point congruence for actions of nontrivial p-groups,

$$|G/H| \equiv |\operatorname{Fix}_H(G/H)| \bmod p. \tag{2.1}$$

Let's unravel what it means for a coset gH in G/H to be a fixed point by the group H under left multiplication:

$$\begin{aligned} hgH = gH \text{ for all } h \in H &\iff hg \in gH \text{ for all } h \in H \\ &\iff g^{-1}hg \in H \text{ for all } h \in H \\ &\iff g^{-1}Hg \subset H \\ &\iff g^{-1}Hg = H \text{ because } |g^{-1}Hg| = |H| \\ &\iff g \in \mathrm{N}(H). \end{aligned}$$

Thus $\operatorname{Fix}_H(G/H) = \{gH : g \in \mathrm{N}(H)\} = \mathrm{N}(H)/H$, so (2.1) becomes

$$[G : H] \equiv [\mathrm{N}(H) : H] \bmod p. \tag{2.2}$$

Because $H \lhd \mathrm{N}(H)$, $\mathrm{N}(H)/H$ *is a group.*

When $|H| = p^i$ and $i < k$, the index $[G : H]$ is divisible by p, so the congruence (2.2) implies $[\mathrm{N}(H) : H]$ is divisible by p, so $\mathrm{N}(H)/H$ is a group with order divisible by p. Thus $\mathrm{N}(H)/H$ has a subgroup of order p by Cauchy's theorem. All subgroups of the quotient group $\mathrm{N}(H)/H$ have the form H'/H, where H' is a subgroup between H and $\mathrm{N}(H)$. Therefore a subgroup of order p in $\mathrm{N}(H)/H$ is H'/H such that $[H' : H] = p$, so $|H'| = p|H| = p^{i+1}$.

Proof of Sylow II: Pick two p-Sylow subgroups P and Q. We want to show they are conjugate.

Consider the action of Q on G/P by left multiplication. Since Q is a finite p-group,

$$|G/P| \equiv |\operatorname{Fix}_Q(G/P)| \bmod p.$$

The left side is $[G : P]$, which is nonzero modulo p since P is a p-Sylow subgroup. Thus $|\operatorname{Fix}_Q(G/P)|$ can't be 0, so there is a fixed point in G/P. Call it gP. That is, $qgP = gP$ for all $q \in Q$. Equivalently, $qg \in gP$ for all $q \in Q$, so $Q \subset gPg^{-1}$. Therefore $Q = gPg^{-1}$, since Q and gPg^{-1} have the same size.

Proof of Sylow III: We will prove $n_p \equiv 1 \bmod p$ and then $n_p \mid m$.

To show $n_p \equiv 1 \bmod p$, consider the action of P on the set $\operatorname{Syl}_p(G)$ by conjugation. The size of $\operatorname{Syl}_p(G)$ is n_p. Since P is a finite p-group,

$$n_p \equiv |\{\text{fixed points}\}| \bmod p.$$

Fixed points for P acting by conjugation on $\operatorname{Syl}_p(G)$ are $Q \in \operatorname{Syl}_p(G)$ such that $gQg^{-1} = Q$ for all $g \in P$. One choice for Q is P. For any such Q, $P \subset \mathrm{N}(Q)$. Also $Q \subset \mathrm{N}(Q)$, so P and Q are p-Sylow subgroups in $\mathrm{N}(Q)$. Applying Sylow II to the group $\mathrm{N}(Q)$, P and Q are conjugate in $\mathrm{N}(Q)$. Since $Q \lhd \mathrm{N}(Q)$, the only subgroup of $\mathrm{N}(Q)$ conjugate to Q is Q, so $P = Q$. Thus P is the only fixed point when P acts on $\operatorname{Syl}_p(G)$, so $n_p \equiv 1 \bmod p$.

To show $n_p \mid m$, consider the action of G by conjugation on $\operatorname{Syl}_p(G)$. Since the p-Sylow subgroups are conjugate to each other (Sylow II), there is one orbit. A set on which a group acts with one orbit has size dividing the size of the group, so $n_p \mid |G|$. From $n_p \equiv 1 \bmod p$, the number n_p is relatively prime to p, so $n_p \mid m$.

Proof of Sylow III*: Let P be a p-Sylow subgroup of G and let G act on $\operatorname{Syl}_p(G)$ by conjugation. By the orbit-stabilizer formula,

$$n_p = |\operatorname{Syl}_p(G)| = [G : \operatorname{Stab}_{\{P\}}].$$

The stabilizer $\operatorname{Stab}_{\{P\}}$ is

$$\operatorname{Stab}_{\{P\}} = \{g : gPg^{-1} = P\} = \mathrm{N}(P).$$

Thus $n_p = [G : \mathrm{N}(P)]$.

3. Historical Remarks

Sylow's proof of his theorems appeared in [1]. Here is what he showed (of course, without using the label "Sylow subgroup").

1) There exist p-Sylow subgroups. Moreover, $[G : \mathrm{N}(P)] \equiv 1 \bmod p$ for any p-Sylow subgroup P.
2) Let P be a p-Sylow subgroup. The number of p-Sylow subgroups is $[G : \mathrm{N}(P)]$. All p-Sylow subgroups are conjugate.
3) Any finite p-group G with size p^k contains an increasing chain of subgroups

$$\{e\} = G_0 \subset G_1 \subset G_2 \subset \cdots \subset G_k \subset G,$$

where each subgroup has index p in the next one. In particular, $|G_i| = p^i$ for all i.

Here is how Sylow phrased his first theorem (the first item on the above list):[1]

[1]We modify some of his notation: he wrote the subgroup as g, not H, and the prime as n, not p.

> Si p^α désigne la plus grande puissance du nombre premier p qui divise l'ordre du groupe G, ce groupe contient un autre H de l'ordre p^α; si de plus $p^\alpha\nu$ désigne l'ordre du plus grand groupe contenu dans G dont les substitutions sont permutables à H, l'ordre de G sera de la forme $p^\alpha\nu(pm+1)$.

In English, using current terminology, this says

> If p^α is the largest power of the prime p which divides the size of the group G, this group contains a subgroup H of order p^α; if moreover $p^\alpha\nu$ is the size of the largest subgroup of G that normalizes H, the size of G is of the form $p^\alpha\nu(pm+1)$.

Sylow did not have the abstract concept of a group: all groups for him arose as subgroups of symmetric groups, so groups were always "groupes de substitutions." The condition that an element $x \in G$ is "permutable" with a subgroup H means $xH = Hx$, or in other words $x \in \mathrm{N}(H)$. The end of the first part of his theorem says the normalizer of a Sylow subgroup has index $pm + 1$ for some m, which means the index is $\equiv 1 \bmod p$.

4. Analogues of the Sylow Theorems

There are analogues of the Sylow theorems for other types of subgroups.

(1) A *Hall subgroup* of a finite group G is a subgroup H whose order and index are relatively prime. For example, in a group of order 60 any subgroup of order 12 has index 5 and thus is a Hall subgroup. A p-subgroup is a Hall subgroup if and only if it is a Sylow subgroup. In 1928 Philip Hall proved in every solvable group of order n that there is a Hall subgroup of each order d dividing n where $(d, n/d) = 1$ and any two Hall subgroups with the same order are conjugate. Conversely, Hall proved that a finite group of order n containing a Hall subgroup of order d for each d dividing n such that $(d, n/d) = 1$ has to be a solvable group.

(2) In a compact connected Lie group, the *maximal tori* (maximal connected abelian subgroups) satisfy properties analogous to Sylow subgroups: they exist, every torus is contained in a maximal torus, and all maximal tori are conjugate. Of course, unlike Sylow subgroups, maximal tori are always abelian.

(3) In a connected linear algebraic group, the *maximal unipotent subgroups* are like Sylow subgroups: they exist, every unipotent subgroup is contained in a maximal unipotent subgroup, and all maximal unipotent subgroups are conjugate. The normalizer of a maximal unipotent subgroup is called a Borel subgroup, and like the normalizers of Sylow subgroups all Borel subgroups equal their own normalizer. For the group $\mathrm{GL}_n(\mathbf{Z}/(p))$, its subgroup of upper triangular matrices with 1's along the main diagonal

$$\begin{pmatrix} 1 & * & \cdots & * \\ 0 & 1 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

is both a p-Sylow subgroup and a maximal unipotent subgroup.

References

[1] L. Sylow, Théorèmes sur les groupes de substitutions, *Mathematische Annalen* **5** (1872), 584–594. Translation into English by Robert Wilson at http://www.maths.qmul.ac.uk/~raw/pubs_files/Sylow.pdf.