SPLITTING OF SHORT EXACT SEQUENCES FOR GROUPS

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1. Introduction

A sequence of groups and group homomorphisms

$$H \xrightarrow{\alpha} G \xrightarrow{\beta} K$$

is called *exact* at G if $\operatorname{im}\alpha = \ker\beta$. This means two things: the image of α is killed by β ($\beta(\alpha(h)) = 1$ for all $h \in H$), so $\operatorname{im}\alpha \subset \ker\beta$, and also *only* the image of α is killed by β (if $\beta(g) = 1$ then $g = \alpha(h)$ for some h), so $\ker\beta \subset \operatorname{im}\alpha$. For example, to say $1 \longrightarrow G \xrightarrow{f} K$ is exact at G means f is injective, and to say $H \xrightarrow{f} G \longrightarrow 1$ is exact at G means f is surjective. There is no need to label the homomorphisms coming out of 1 or going to 1 since there is only one possible choice. If the group operations are written additively, we may use 0 in place of 1 for the trivial group.

A *short exact sequence* of groups is a sequence of groups and group homomorphisms

$$1 \longrightarrow H \xrightarrow{\alpha} G \xrightarrow{\beta} K \longrightarrow 1 \tag{1.1}$$

which is exact at H, G, and K. That means α is injective, β is surjective, and $\operatorname{im}\alpha = \ker\beta$.

A more general exact sequence can have lots of terms:

$$G_1 \xrightarrow{\alpha_1} G_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} G_n, \tag{1.2}$$

and it must be exact at each G_i for $1 < i < n$. Exact sequences can also be of infinite length in one or both directions. We will only deal with short exact sequences here.

Exact sequences first arose in algebraic topology, and the later development of homological algebra (the type of algebra underlying algebraic topology) spread exact sequences into the rest of mathematics.

Example 1.1. The determinant on $\mathrm{GL}_2(\mathbf{R})$ gives rise to a short exact sequence

$$1 \longrightarrow \mathrm{SL}_2(\mathbf{R}) \longrightarrow \mathrm{GL}_2(\mathbf{R}) \xrightarrow{\det} \mathbf{R}^\times \longrightarrow 1.$$

Example 1.2. When $N \lhd G$ we have a short exact sequence

$$1 \longrightarrow N \longrightarrow G \longrightarrow G/N \longrightarrow 1, \tag{1.3}$$

where the map from N to G is inclusion, and the map from G to G/N is reduction mod N.

This example is the prototype for all short exact sequences, as we'll see below.

Example 1.3. For two groups H and K, the direct product $H \times K$ fits into the short exact sequence

$$1 \longrightarrow H \longrightarrow H \times K \longrightarrow K \longrightarrow 1,$$

where the map out of H is embedding to the first factor ($h \mapsto (h, 1)$) and the map out of $H \times K$ is projection to the second factor ($(h, k) \mapsto k$).

Example 1.4. For two groups H and K, together with an action of K on H by automorphisms (a homomorphism $\varphi\colon K \to \mathrm{Aut}(H)$), the semidirect product $H \rtimes_\varphi K$ fits into the short exact sequence

$$1 \longrightarrow H \longrightarrow H \rtimes_\varphi K \longrightarrow K \longrightarrow 1,$$

where the maps are the same as in the previous example: $h \mapsto (h, 1)$ and $(h, k) \mapsto k$.

Every short exact sequence (1.1) is a disguised form of (1.3). Indeed, even though in (1.1) the group H may not literally be a subgroup of G and the group K may not literally be a quotient group of G, α restricts to an isomorphism of H with the subgroup $\alpha(H)(= \ker\beta)$ of G and β induces an isomorphism $\overline{\beta}$ of the quotient group $G/\alpha(H) = G/\ker\beta$ with K. Therefore we can place the general short exact sequence (1.1) and a short exact sequence of the type (1.3) in a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & H \arrow[r, "\alpha"] \arrow[d, "\alpha"'] & G \arrow[r, "\beta"] \arrow[d, "\mathrm{id}"'] & K \arrow[r] \arrow[d, "\overline{\beta}"'] & 1 \\
1 \arrow[r] & \alpha(H) \arrow[r] & G \arrow[r] & G/\ker\beta \arrow[r] & 1
\end{tikzcd} \tag{1.4}$$

where the bottom short exact sequence is a special case of (1.3). The vertical maps are all isomorphisms, and in this sense (1.1) looks like (1.3): they are linked to each other through compatible isomorphisms of groups in the same positions in the two short exact sequences. (The compatibility of the isomorphisms simply means the diagram (1.4) commutes.)

In Section 2 we will look at some more examples of short exact sequences. Then in Section 3, which is the most important part, we will see how direct products and semidirect products of groups can be characterized in terms of short exact sequences with extra structure. Section 4 discusses the idea of two short exact sequences being alike in broad terms.

2. Examples

When $N \lhd G$, knowing N and G/N does not usually tell us what G is. That is, nonisomorphic groups can have isomorphic normal subgroups with isomorphic quotient groups. For example, $D_4 \not\cong Q_8$ but $\langle r^2\rangle \cong \{\pm 1\}(\cong \mathbf{Z}/2\mathbf{Z})$ and $D_4/\langle r^2\rangle \cong Q_8/\{\pm 1\}(\cong (\mathbf{Z}/2\mathbf{Z})^2)$. In terms of short exact sequences, the two short exact sequences

$$1 \longrightarrow \langle r^2\rangle \longrightarrow D_4 \longrightarrow D_4/\langle r^2\rangle \longrightarrow 1$$

and

$$1 \longrightarrow \{\pm 1\} \longrightarrow Q_8 \longrightarrow Q_8/\{\pm 1\} \longrightarrow 1 \tag{2.1}$$

have isomorphic first groups and isomorphic third groups, but nonisomorphic middle groups. Here is a third example like these, with an abelian group in the middle:

$$0 \longrightarrow \mathbf{Z}/2\mathbf{Z} \longrightarrow \mathbf{Z}/2\mathbf{Z} \times (\mathbf{Z}/2\mathbf{Z})^2 \longrightarrow (\mathbf{Z}/2\mathbf{Z})^2 \longrightarrow 0.$$

This is the short exact sequence for a direct product, as in Example 1.3.

Here are two examples of short exact sequences with first group $\mathbf{Z}/4\mathbf{Z}$ and third group $\mathbf{Z}/2\mathbf{Z}$, but nonisomorphic groups in the middle:

$$0 \longrightarrow \mathbf{Z}/4\mathbf{Z} \longrightarrow \mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z} \longrightarrow \mathbf{Z}/2\mathbf{Z} \longrightarrow 0$$
$$0 \longrightarrow \mathbf{Z}/4\mathbf{Z} \longrightarrow \mathbf{Z}/8\mathbf{Z} \longrightarrow \mathbf{Z}/2\mathbf{Z} \longrightarrow 0,$$

where the map $\mathbf{Z}/4\mathbf{Z} \to \mathbf{Z}/8\mathbf{Z}$ in the second short exact sequence is doubling ($x \bmod 4 \mapsto 2x \bmod 8$). The other maps are all the obvious ones.

Here are two short exact sequences with first and third groups equal to μ_m ($m > 1$) but nonisomorphic groups in the middle:

$$1 \longrightarrow \mu_m \longrightarrow \mu_m \times \mu_m \longrightarrow \mu_m \longrightarrow 1$$

and

$$1 \longrightarrow \mu_m \xrightarrow{\iota} \mu_{m^2} \xrightarrow{z \mapsto z^m} \mu_m \longrightarrow 1.$$

The first short exact sequence is the usual one for a direct product. In the second short exact sequence, ι is the inclusion. The middle groups $\mu_m \times \mu_m$ and μ_{m^2} are not isomorphic since $\mu_m \times \mu_m$ is not cyclic (no element of order m^2).

3. Direct and Semidirect Products

For two groups H and K, an important "lifting" problem is the determination of all groups G having a normal subgroup isomorphic to H and corresponding quotient group isomorphic to K. Such G are the groups that fit into a short exact sequence $1 \longrightarrow H \longrightarrow G \longrightarrow K \longrightarrow 1$. There is always at least one such G, namely $H \times K$. More generally, a semidirect product $H \rtimes_\varphi K$ always sits in a short exact sequence having kernel H and image K (Example 1.4). Not all short exact sequences arise from semidirect products.

Example 3.1. In the short exact sequence (2.1), Q_8 is not isomorphic to a semidirect product of $\{\pm 1\}$ and $Q_8/\{\pm 1\} \cong (\mathbf{Z}/2\mathbf{Z})^2$ since such a semidirect product has more than 1 element of order 2 while Q_8 has only one element of order 2.

Since semidirect products are "known," short exact sequences made with them are considered "trivial" (even though semidirect products may seem like a nontrivial way to create groups). It is important to recognize if a short exact sequence $1 \longrightarrow H \xrightarrow{\alpha} G \xrightarrow{\beta} K \longrightarrow 1$ is essentially that for a direct product $H \times K$ or semidirect product $H \rtimes_\varphi K$. The next two theorems give such criteria in terms of a left inverse for α and a right inverse for β.

Theorem 3.2. *Let $1 \longrightarrow H \xrightarrow{\alpha} G \xrightarrow{\beta} K \longrightarrow 1$ be a short exact sequence of groups. The following are equivalent:*

(1) *There is a homomorphism $\alpha' \colon G \to H$ such that $\alpha'(\alpha(h)) = h$ for all $h \in H$.*
(2) *There is an isomorphism $\theta \colon G \to H \times K$ such that the diagram*

$$\begin{tikzcd} 1 \arrow[r] & H \arrow[r, "\alpha"] \arrow[d, "\mathrm{id}"'] & G \arrow[r, "\beta"] \arrow[d, "\theta"'] & K \arrow[r] \arrow[d, "\mathrm{id}"'] & 1 \\ 1 \arrow[r] & H \arrow[r] & H \times K \arrow[r] & K \arrow[r] & 1 \end{tikzcd}$$

commutes, where the bottom row is the short exact sequence for a direct product.

The commutative diagram in (2) says that θ identifies α with the embedding $H \to H \times K$ and β with the projection $H \times K \to K$. So the point of (2) is not simply that G is isomorphic to $H \times K$, but it is in a way that turns α and β into the standard maps from H to $H \times K$ and from $H \times K$ to K.

The key point of (1) is that α' is a homomorphism. Merely from α being injective, there is a *function* $\alpha' \colon G \to H$ such that $\alpha'(\alpha(h)) = h$ for all h, for instance the function

$$\alpha'(g) = \begin{cases} 1, & \text{if } g \notin \alpha(H), \\ h, & \text{if } g = \alpha(h). \end{cases}$$

But this α' is almost surely not a homomorphism.

Proof. (1) $\Rightarrow$ (2): Define $\theta\colon G \to H \times K$ by

$$\theta(g) = (\alpha'(g), \beta(g)).$$

This is a homomorphism since α' and β are homomorphisms. To see θ is injective, suppose $\theta(g) = (1,1)$, so $\alpha'(g) = 1$ and $\beta(g) = 1$. From exactness at G, the condition $\beta(g) = 1$ implies $g = \alpha(h)$ for some $h \in H$. Then $1 = \alpha'(g) = \alpha'(\alpha(h)) = h$, so $g = \alpha(h) = \alpha(1) = 1$.

To show θ is surjective, let $(h,k) \in H \times K$. Since β is onto, $k = \beta(g)$ for some $g \in G$. Since $\ker\beta = \operatorname{im}\alpha$, the general inverse image of k under β is $g\alpha(x)$ for $x \in H$. We want to find $x \in H$ such that $\alpha'(g\alpha(x)) = h$, so then $\theta(g\alpha(x)) = (h,k)$. Since α' is a homomorphism, the condition $\alpha'(g\alpha(x)) = h$ is equivalent to $\alpha'(g)x = h$, so *define* $x = \alpha'(g)^{-1}h$. Then

$$\theta(g\alpha(x)) = (\alpha'(g\alpha(x)), \beta(g\alpha(x))) = (h,k),$$

so θ is an isomorphism from G to $H \times K$.

Next, we want to check the diagram

$$\begin{tikzcd} 1 \arrow[r] & H \arrow[r, "\alpha"] \arrow[d, "\mathrm{id}"'] & G \arrow[r, "\beta"] \arrow[d, "\theta"'] & K \arrow[r] \arrow[d, "\mathrm{id}"'] & 1 \\ 1 \arrow[r] & H \arrow[r] & H \times K \arrow[r] & K \arrow[r] & 1 \end{tikzcd}$$

commutes. In the first square

$$\begin{tikzcd} H \arrow[r, "\alpha"] \arrow[d, "\mathrm{id}"'] & G \arrow[d, "\theta"'] \\ H \arrow[r] & H \times K \end{tikzcd}$$

taking $h \in H$ along the top and right has the effect $h \mapsto \alpha(h) \mapsto (\alpha'(\alpha(h)), \beta(\alpha(h))) = (h,1)$, which is also the result of taking h along the left and bottom. In the second square

$$\begin{tikzcd} G \arrow[r, "\beta"] \arrow[d, "\theta"'] & K \arrow[d, "\mathrm{id}"'] \\ H \times K \arrow[r] & K \end{tikzcd}$$

taking $g \in G$ along the top and right has the effect $g \mapsto \beta(g) \mapsto \beta(g)$, and going along the left and bottom leads to $g \mapsto (\alpha'(g), \beta(g)) \mapsto \beta(g)$. So the diagram commutes.

(2) $\Rightarrow$ (1): Suppose there is an isomorphism $\theta\colon G \to H \times K$ such that

$$\begin{tikzcd} 1 \arrow[r] & H \arrow[r, "\alpha"] \arrow[d, "\mathrm{id}"'] & G \arrow[r, "\beta"] \arrow[d, "\theta"'] & K \arrow[r] \arrow[d, "\mathrm{id}"'] & 1 \\ 1 \arrow[r] & H \arrow[r] & H \times K \arrow[r] & K \arrow[r] & 1 \end{tikzcd}$$

commutes. For $g \in G$, $\theta(g) \in H \times K$ has second coordinate $\beta(g)$ from commutativity of the second square. Let $\alpha'(g)$ denote the first coordinate:

$$\theta(g) = (\alpha'(g), \beta(g)).$$

Then $\alpha' \colon G \to H$ is a function and θ is a homomorphism, so α' is a homomorphism. The commutativity of the first square implies $\theta(\alpha(h)) = (h, 1)$, so $(\alpha'(\alpha(h)), \beta(\alpha(h))) = (h, 1)$, so $\alpha'(\alpha(h)) = h$ for all $h \in H$. $\square$

The proof shows not only that conditions (1) and (2) are equivalent, but that the homomorphisms α' in (1) and isomorphisms θ in (2) are in bijection from the formula $\theta(g) = (\alpha'(g), \beta(g))$.

Theorem 3.3. *Let* $1 \longrightarrow H \xrightarrow{\alpha} G \xrightarrow{\beta} K \longrightarrow 1$ *be a short exact sequence. The following are equivalent:*

(1) *There is a homomorphism* $\beta' \colon K \to G$ *such that* $\beta(\beta'(k)) = k$ *for all* $k \in K$*.*
(2) *There is a homomorphism* $\varphi \colon K \to \operatorname{Aut}(H)$ *and an isomorphism* $\theta \colon G \to H \rtimes_\varphi K$ *such that the diagram*

$$\begin{tikzcd} 1 \arrow[r] & H \arrow[r, "\alpha"] \arrow[d, "\mathrm{id}"'] & G \arrow[r, "\beta"] \arrow[d, "\theta"'] & K \arrow[r] \arrow[d, "\mathrm{id}"'] & 1 \\ 1 \arrow[r] & H \arrow[r] & H \rtimes_\varphi K \arrow[r] & K \arrow[r] & 1 \end{tikzcd}$$

commutes, where the bottom short exact sequence is the usual one for a semidirect product.

As with Theorem 3.2, the key part of (1) is that β' is a homomorphism. From β being surjective, there is a function $\beta' \colon K \to G$ such that $\beta(\beta'(k)) = k$ for all $k \in K$, for instance set $\beta'(k)$ for each k to be a solution to $\beta(g) = k$. But usually this β' is not a *homomorphism.*

Proof. (1) $\Rightarrow$ (2): From the homomorphism β' we have to create an action φ of K on H by automorphisms and an isomorphism of G with $H \rtimes_\varphi K$. Using β', we can make K act on H using conjugation in G: for $k \in K$ and $h \in H$, $\beta'(k)\alpha(h)\beta'(k^{-1}) \in \ker \beta$ since

$$\beta(\beta'(k)\alpha(h)\beta'(k^{-1})) = \beta(\beta'(k))\beta(\alpha(h))\beta(\beta'(k^{-1})) = k \cdot 1 \cdot k^{-1} = 1.$$

Since $\ker \beta = \operatorname{im} \alpha$, we can write $\beta'(k)\alpha(h)\beta'(k^{-1}) = \alpha(h')$ for some $h' \in H$, and h' is unique since α is injective. Set $\varphi_k(h) = h'$, so $\varphi_k(h)$ is the unique element of H such that

$$\beta'(k)\alpha(h)\beta'(k)^{-1} = \alpha(\varphi_k(h)), \tag{3.1}$$

where we write $\beta'(k^{-1})$ as $\beta'(k)^{-1}$ since β' is a homomorphism. We want to show $k \mapsto \varphi_k$ is a homomorphism from K to $\operatorname{Aut}(H)$.

First, setting $k = 1$ in (3.1), $\alpha(h) = \alpha(\varphi_1(h))$, so $\varphi_1(h) = h$ for all $h \in H$. Thus $\varphi_1 = \mathrm{id}_H$. Next we check $\varphi_k \colon H \to H$ is a homomorphism for each $k \in K$. For h_1 and h_2 in H, $\varphi_k(h_1h_2)$ is characterized by the equation $\beta'(k)\alpha(h_1h_2)\beta'(k)^{-1} = \alpha(\varphi_k(h_1h_2))$. The left side is

$$\begin{aligned} \beta'(k)\alpha(h_1)\alpha(h_2)\beta'(k)^{-1} &= \beta'(k)\alpha(h_1)\beta'(k)^{-1}\beta'(k)\alpha(h_2)\beta'(k)^{-1} \\ &= \alpha(\varphi_k(h_1))\alpha(\varphi_k(h_2)) \\ &= \alpha(\varphi_k(h_1)\varphi_k(h_2)), \end{aligned}$$

so by injectivity of α we have $\varphi_k(h_1)\varphi_k(h_2) = \varphi_k(h_1h_2)$.

Next we show $\varphi_{k_1} \circ \varphi_{k_2} = \varphi_{k_1k_2}$. For $h \in H$, $\varphi_{k_1k_2}(h)$ is characterized by the equation

$$\beta'(k_1k_2)\alpha(h)\beta'(k_1k_2)^{-1} = \alpha(\varphi_{k_1k_2}(h)),$$

and the left side is

$$\begin{aligned}\beta'(k_1)\beta'(k_2)\alpha(h)\beta'(k_2)^{-1}\beta'(k_1)^{-1} &= \beta'(k_1)\alpha(\varphi_{k_2}(h))\beta'(k_1)^{-1} \quad \text{by (3.1)}\\ &= \alpha(\varphi_{k_1}(\varphi_{k_2}(h))),\end{aligned}$$

so $\varphi_{k_1}(\varphi_{k_2}(h)) = \varphi_{k_1k_2}(h)$, so $\varphi_{k_1} \circ \varphi_{k_2} = \varphi_{k_1k_2}$. In particular, $\varphi_k \circ \varphi_{k^{-1}} = \varphi_1$ and $\varphi_{k^{-1}} \circ \varphi_k = \varphi_1$, so $\varphi_k \in \mathrm{Aut}(H)$ and $k \mapsto \varphi_k$ is a homomorphism $K \to \mathrm{Aut}(H)$. We have proved that (3.1) provides an action of K on H by automorphisms, so we have the semidirect product $H \rtimes_\varphi K$.

To get an isomorphism $G \to H \rtimes_\varphi K$, it is easier to go in the other direction. Let $\gamma\colon H \rtimes_\varphi K \to G$ by

$$\gamma(h,k) = \alpha(h)\beta'(k).$$

To check γ is a homomorphism,

$$\begin{aligned}\gamma((h_1,k_1)(h_2,k_2)) &= \gamma(h_1\varphi_{k_1}(h_2), k_1k_2)\\ &= \alpha(h_1\varphi_{k_1}(h_2))\beta'(k_1k_2)\\ &= \alpha(h_1)\alpha(\varphi_{k_1}(h_2))\beta'(k_1)\beta'(k_2)\\ &= \alpha(h_1)(\beta'(k_1)\alpha(h_2)\beta'(k_1)^{-1})\beta'(k_1)\beta'(k_2) \quad \text{by (3.1)}\\ &= \alpha(h_1)\beta'(k_1)\alpha(h_2)\beta'(k_2)\\ &= \gamma(h_1,k_1)\gamma(h_2,k_2).\end{aligned}$$

To show γ is injective, if $\gamma(h,k) = 1$ then $\alpha(h)\beta'(k) = 1$. Applying β to both sides, $\beta(\alpha(h))\beta(\beta'(k)) = \beta(1) = 1$, so $k = 1$. Then $\alpha(h) \cdot 1 = 1$, so $h = 1$ since α is injective.

To show γ is surjective, pick $g \in G$. We want to find $h \in H$ and $k \in K$ such that $\alpha(h)\beta'(k) = g$. Applying β to both sides, $\beta(\alpha(h))\beta(\beta'(k)) = \beta(g)$, so $k = \beta(g)$. So we *define* $k := \beta(g)$ and then ask if there is $h \in H$ such that $\alpha(h) = g\beta'(k^{-1}) = g\beta'(\beta(g)^{-1})$. Since $\operatorname{im}\alpha = \ker\beta$, whether or not there is such an h is equivalent to checking $g\beta'(\beta(g)^{-1})) \in \ker\beta$:

$$\begin{aligned}\beta(g\beta'(\beta(g)^{-1})) &= \beta(g)\beta(\beta'(\beta(g)^{-1}))\\ &= \beta(g)\beta(g)^{-1}\\ &= 1.\end{aligned}$$

Thus $\gamma\colon H \rtimes_\varphi K \to G$ is an isomorphism. Let $\theta = \gamma^{-1}$ be the inverse isomorphism.

Finally, to show the diagram

$$\begin{tikzcd} 1 \arrow[r] & H \arrow[r, "\alpha"] \arrow[d, "\mathrm{id}"'] & G \arrow[r, "\beta"] \arrow[d, "\theta"'] & K \arrow[r] \arrow[d, "\mathrm{id}"'] & 1 \\ 1 \arrow[r] & H \arrow[r] & H \rtimes_\varphi K \arrow[r] & K \arrow[r] & 1 \end{tikzcd}$$

commutes, it is equivalent to show the "flipped" diagram

$$\begin{tikzcd} 1 \arrow[r] & H \arrow[r, "\alpha"] & G \arrow[r, "\beta"] & K \arrow[r] & 1 \\ 1 \arrow[r] & H \arrow[r] \arrow[u, "\mathrm{id}"'] & H \rtimes_\varphi K \arrow[r] \arrow[u, "\gamma"'] & K \arrow[r] \arrow[u, "\mathrm{id}"'] & 1 \end{tikzcd}$$

commutes. For $h \in H$, going around the first square along the left and top has the effect $h \mapsto h \mapsto \alpha(h)$, and going around the other way has the effect $h \mapsto (h,1) \mapsto \gamma(h,1) =$

$\alpha(h)\beta'(1) = \alpha(h)$. In the second square, for $(h, k) \in H \rtimes_\varphi K$ going around the left and top has the effect $(h, k) \mapsto \beta(\gamma(h, k)) = \beta(\alpha(h))\beta(\beta'(k)) = k$, while going around the other way has the effect $(h, k) \mapsto k \mapsto k$.

(2) $\Rightarrow$ (1): In the proof that (1) $\Rightarrow$ (2), $\gamma(h, k) = \alpha(h)\beta'(k)$, so $\gamma(1, k) = \beta'(k)$. This *suggests* that when we have an isomorphism $\theta\colon G \to H \rtimes_\varphi K$ that we define $\beta'\colon K \to G$ by $\beta'(k) = \theta^{-1}(1, k)$. This is a homomorphism since $k \mapsto (1, k)$ is a homomorphism and θ^{-1} is a homomorphism. The composite $\beta(\beta'(k)) = \beta(\theta^{-1}(1, k))$ equals k from commutativity of the diagram

$$\begin{tikzcd} G \arrow[r, "\beta"] & K \\ H \rtimes_\varphi K \arrow[u, "\theta^{-1}"] \arrow[r] & K. \arrow[u, "\mathrm{id}"'] \end{tikzcd}$$

□

Definition 3.4. A short exact sequence $1 \longrightarrow H \xrightarrow{\alpha} G \xrightarrow{\beta} K \longrightarrow 1$ is said to *split* if it fits the conditions of Theorem 3.3.

A split short exact sequence is one that essentially corresponds to the standard short exact sequence for a semidirect product. One nonsplit short exact sequence is (2.1).

Since a semidirect product is usually not a direct product, the first conditions in Theorems 3.2 and 3.3 are not equivalent: for a short exact sequence of groups $1 \longrightarrow H \xrightarrow{\alpha} G \xrightarrow{\beta} K \longrightarrow 1$, if there is a homomorphism $\beta'\colon K \to G$ such that $\beta(\beta'(k)) = k$ for all k there need not be a homomorphism $\alpha'\colon G \to H$ such that $\alpha'(\alpha(h)) = h$ for all h. However, when G is abelian, (3.1) simplifies to $\alpha(h) = \alpha(\varphi_k(h))$ for all k and h, so $h = \varphi_k(h)$. Thus K acts trivially on H, so $H \rtimes_\varphi K = H \times K$. Therefore Theorems 3.2 and 3.3 provide three equivalent conditions on $1 \longrightarrow H \xrightarrow{\alpha} G \xrightarrow{\beta} K \longrightarrow 1$ when G is abelian:

(1) There is a homomorphism $\alpha'\colon G \to H$ such that $\alpha'(\alpha(h)) = h$ for all $h \in H$.
(2) There is a homomorphism $\beta'\colon K \to G$ such that $\beta(\beta'(k)) = k$ for all $k \in K$.
(3) There is an isomorphism $\theta\colon G \to H \times K$ such that the diagram

$$\begin{tikzcd} 1 \arrow[r] & H \arrow[r, "\alpha"] \arrow[d, "\mathrm{id}"'] & G \arrow[r, "\beta"] \arrow[d, "\theta"'] & K \arrow[r] \arrow[d, "\mathrm{id}"'] & 1 \\ 1 \arrow[r] & H \arrow[r] & H \times K \arrow[r] & K \arrow[r] & 1 \end{tikzcd}$$

commutes, where the bottom row is the short exact sequence for a direct product.

4. Equivalent Short Exact Sequences

We said in the introduction that every short exact sequence (1.1) is basically like a short exact sequence of type (1.3), and made the idea precise in terms of a commutative diagram (1.4) having both short exact sequences (1.1) and (1.3) appearing in it as the rows, and the columns being isomorphisms. This idea of two short exact sequences being basically alike can be applied more generally. Say $1 \longrightarrow H_1 \xrightarrow{\alpha_1} G_1 \xrightarrow{\beta_1} K_2 \longrightarrow 1$ and

$1 \longrightarrow H_2 \xrightarrow{\alpha_2} G_2 \xrightarrow{\beta_2} K_2 \longrightarrow 1$ are *equivalent* if they fit into a commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & H_1 \arrow[r, "\alpha_1"] \arrow[d] & G_1 \arrow[r, "\beta_1"] \arrow[d] & K_1 \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & H_2 \arrow[r, "\alpha_2"] & G_2 \arrow[r, "\beta_2"] & K_2 \arrow[r] & 1 \end{tikzcd} \tag{4.1}$$

where the vertical maps are isomorphisms. In this terminology, (1.4) shows every short exact sequence is equivalent to a short exact sequence of type (1.3). When two short exact sequences are equivalent, the first groups both sit inside the second groups in the same way, and the third groups are homomorphic images of the second groups in the same way.

Here is a concrete example of equivalent short exact sequences:

$$1 \longrightarrow A_3 \longrightarrow S_3 \xrightarrow{\text{sgn}} \{\pm 1\} \longrightarrow 1 \tag{4.2}$$

and

$$0 \longrightarrow \mathbf{Z}/3\mathbf{Z} \longrightarrow \text{Aff}(\mathbf{Z}/3\mathbf{Z}) \xrightarrow{\det} (\mathbf{Z}/3\mathbf{Z})^\times \longrightarrow 1, \tag{4.3}$$

where the first map in (4.2) is inclusion and the first map in (4.3) is $b \mapsto \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$. These are equivalent because there is a commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & A_3 \arrow[r] \arrow[d] & S_3 \arrow[r, "\text{sgn}"] \arrow[d] & \{\pm 1\} \arrow[r] \arrow[d] & 1 \\ 0 \arrow[r] & \mathbf{Z}/3\mathbf{Z} \arrow[r] & \text{Aff}(\mathbf{Z}/3\mathbf{Z}) \arrow[r, "\det"] & (\mathbf{Z}/3\mathbf{Z})^\times \arrow[r] & 1 \end{tikzcd}$$

where (4.2) and (4.3) are the rows and the vertical maps are all isomorphisms.

Theorem 3.2 gives us a condition for detecting when a short exact sequence $1 \longrightarrow H \xrightarrow{\alpha} G \xrightarrow{\beta} K \longrightarrow 1$ is equivalent to the usual short exact sequence for $H \times K$ with the first and third vertical maps being the identities on H and K. Similarly, Theorem 3.3 tells us when $1 \longrightarrow H \xrightarrow{\alpha} G \xrightarrow{\beta} K \longrightarrow 1$ is equivalent to the usual short exact sequence for some semidirect product $H \rtimes_\varphi K$ with the first and third vertical maps being the identities on H and K.

The notion of equivalent short exact sequences is an equivalence relation: any short exact sequence $1 \longrightarrow H \xrightarrow{\alpha} G \xrightarrow{\beta} K \longrightarrow 1$ is equivalent to itself from the commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & H \arrow[r, "\alpha"] \arrow[d, "\text{id}"'] & G \arrow[r, "\beta"] \arrow[d, "\text{id}"'] & K \arrow[r] \arrow[d, "\text{id}"'] & 1 \\ 1 \arrow[r] & H \arrow[r, "\alpha"] & G \arrow[r, "\beta"] & K \arrow[r] & 1 \end{tikzcd} \tag{4.4}$$

and using inverse isomorphisms in the vertical rows of (4.1) gives a commutative diagram where the two rows are interchanged, so the notion of equivalent short exact sequences is symmetric. For transitivity, we can combine two commutative diagrams

$$\begin{tikzcd} 1 \arrow[r] & H_1 \arrow[r, "\alpha_1"] \arrow[d] & G_1 \arrow[r, "\beta_1"] \arrow[d] & K_1 \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & H_2 \arrow[r, "\alpha_2"] & G_2 \arrow[r, "\beta_2"] & K_2 \arrow[r] & 1 \end{tikzcd}$$

and

$$\begin{tikzcd} 1 \arrow[r] & H_2 \arrow[r, "\alpha_2"] \arrow[d] & G_2 \arrow[r, "\beta_2"] \arrow[d] & K_2 \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & H_3 \arrow[r, "\alpha_3"] & G_3 \arrow[r, "\beta_3"] & K_3 \arrow[r] & 1 \end{tikzcd}$$

into the commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & H_1 \arrow[r, "\alpha_1"] \arrow[d] & G_1 \arrow[r, "\beta_1"] \arrow[d] & K_1 \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & H_2 \arrow[r, "\alpha_2"] \arrow[d] & G_2 \arrow[r, "\beta_2"] \arrow[d] & K_2 \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & H_3 \arrow[r, "\alpha_3"] & G_3 \arrow[r, "\beta_3"] & K_3 \arrow[r] & 1 \end{tikzcd}$$

and then use the composite of the pairs of vertical isomorphisms to eliminate the middle row and get the commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & H_1 \arrow[r, "\alpha_1"] \arrow[d] & G_1 \arrow[r, "\beta_1"] \arrow[d] & K_1 \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & H_3 \arrow[r, "\alpha_3"] & G_3 \arrow[r, "\beta_3"] & K_3 \arrow[r] & 1 \end{tikzcd}$$

with isomorphisms as the vertical maps.

Here is the analogue of homomorphisms for short exact sequences. A *morphism* from $1 \longrightarrow H_1 \xrightarrow{\alpha_1} G_1 \xrightarrow{\beta_1} K_2 \longrightarrow 1$ to $1 \longrightarrow H_2 \xrightarrow{\alpha_2} G_2 \xrightarrow{\beta_2} K_2 \longrightarrow 1$ is a commutative diagram (4.1) where the vertical maps are homomorphisms rather than isomorphisms. An example of a morphism of short exact sequences is (for $m > 1$)

$$\begin{tikzcd} 1 \arrow[r] & \mathrm{SL}_2(\mathbf{Z}) \arrow[r] \arrow[d] & \mathrm{GL}_2(\mathbf{Z}) \arrow[r, "\det"] \arrow[d] & \{\pm 1\} \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & \mathrm{SL}_2(\mathbf{Z}/m\mathbf{Z}) \arrow[r] & \mathrm{GL}_2(\mathbf{Z}/m\mathbf{Z}) \arrow[r, "\det"] & (\mathbf{Z}/m\mathbf{Z})^\times \arrow[r] & 1, \end{tikzcd}$$

where the vertical maps are the natural mod m reduction maps. The identity morphism for a short exact sequence is (4.4). Our argument that equivalence of short exact sequences is transitive also shows how to compose two morphisms to get a third: compose vertical homomorphisms in the same positions in the two diagrams. What we called equivalence of two short exact sequences is the concept of isomorphism: a morphism of short exact sequences that admits an inverse morphism (one whose composite with the original morphism on both sides gives the identity morphism for the two short exact sequences).