

## MASCHKE'S THEOREM OVER GENERAL FIELDS

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Let  $G$  be a finite group. If  $V$  is a finite-dimensional  $\mathbf{C}$ -vector space on which there is a representation of  $G$ , then Maschke's theorem says every subrepresentation of  $V$  has a complementary subrepresentation: if  $W$  is a  $\mathbf{C}[G]$ -submodule of  $V$  then  $V = W \oplus W'$  for some  $\mathbf{C}[G]$ -submodule  $W'$  of  $V$  (it's not just a subspace:  $G$  carries  $W'$  back to itself). A standard proof of Maschke's theorem uses a  $G$ -invariant Hermitian inner product on  $V$ : if  $\langle \cdot, \cdot \rangle$  is any Hermitian inner product on  $V$  (for example, one defined relative to a choice of basis of  $V$ ) then we can create a  $G$ -invariant Hermitian inner product on  $V$  by averaging:

$$\langle v, v' \rangle_G = \frac{1}{|G|} \sum_{s \in G} \langle s(v), s(v') \rangle.$$

The function  $\langle \cdot, \cdot \rangle_G: V \times V \rightarrow \mathbf{C}$  is easily checked to be a Hermitian inner product on  $V$ . For example  $\langle v, v \rangle_G = (1/|G|) \sum_{s \in G} \langle s(v), s(v) \rangle \geq 0$ , with equality if and only if  $v = 0$  since  $\langle s(v), s(v) \rangle \geq 0$  with equality if and only if  $s(v) = 0$ , which is the same as  $v = 0$ . The function  $\langle \cdot, \cdot \rangle_G$ , by construction, is  $G$ -invariant:  $\langle g(v), g(v') \rangle_G = \langle v, v' \rangle_G$  for all  $v, v' \in V$  and  $g \in G$ . Therefore if  $W$  is a  $\mathbf{C}[G]$ -submodule of  $V$ , its orthogonal subspace

$$W^\perp = \{v \in V : \langle w, v \rangle_G = 0 \text{ for all } w \in W\}$$

is also a  $\mathbf{C}[G]$ -submodule of  $V$  because

$$v \in W^\perp, w \in W, g \in G \Rightarrow \langle w, g(v) \rangle_G = \langle g^{-1}(w), v \rangle_G = 0,$$

the last equality following from  $g^{-1}(w)$  lying in  $W$ . Thus  $g(W^\perp) \subset W^\perp$  for all  $g \in G$ . From the way Hermitian inner products work,  $V = W \oplus W^\perp$ .

Is Maschke's theorem still true if we replace  $\mathbf{C}$  by an arbitrary field  $k$ ? That is, if  $V$  is a finite-dimensional  $k$ -vector space on which there is a representation of  $G$ , does every subrepresentation of  $V$  have a complementary subrepresentation? The averaging technique used above would run into problems if  $|G| = 0$  in  $k$  (which happens if  $k$  has positive characteristic  $p$  and  $|G|$  is divisible by  $p$ ), so we will assume  $|G| \neq 0$  in  $k$ , which is automatically true if  $k$  has characteristic 0. Another problem with extending the proof technique above to general fields is that *Hermitian* inner products rely heavily on  $\mathbf{C}$  having the complex conjugation operation. Replacing the Hermitian inner product on a complex vector space by a nondegenerate bilinear form on a  $k$ -vector space (like the dot product relative to a basis) will not help us, because an orthogonal subspace relative to a nondegenerate bilinear form need not be complementary to the original subspace: necessarily  $\dim(W) + \dim(W^\perp) = \dim(V)$ , but  $W \cap W^\perp$  might be nonzero so we might *not* have  $V = W \oplus W^\perp$ . For example, on  $k^2$  with the nondegenerate bilinear form  $\langle (x, y), (x', y') \rangle = xx' - yy'$ , the subspace  $W = k(1, 1)$  has  $W^\perp = W$ .

It turns out we can prove Maschke's theorem for representations of a finite group over field other than  $\mathbf{C}$  with an averaging trick, but we need to drop the use of bilinear forms to find the complementary subrepresentation.

**Theorem 1.** *Let  $V$  be a finite-dimensional  $k$ -vector space on which there is a representation of a finite group  $G$ . If  $|G| \neq 0$  in  $k$  then each  $k[G]$ -submodule  $W$  of  $V$  has a complementary  $k[G]$ -submodule: there is a  $k[G]$ -submodule  $W'$  such that  $V = W \oplus W'$ .*

*Proof.* The result is obvious if  $W = V$  (use  $W' = \{0\}$ ) or  $W = \{0\}$  (use  $W' = V$ ), so we can assume  $W$  is a nonzero proper subspace of  $V$ . By extending a basis of  $W$  to a basis of  $V$ , we can write  $V = W \oplus \widetilde{W}$  where  $\widetilde{W}$  is the span of the part of that basis of  $V$  not lying in  $W$ . While  $\widetilde{W}$  is a subspace complement of  $W$ , it has no reason to be carried back to itself under the action of  $G$ . Still, it's going to be useful that  $W + \widetilde{W} = V$  and  $W \cap \widetilde{W} = \{0\}$ .

Let  $\text{pr}: V \rightarrow W$  be the projection from  $V$  to  $W$  using the direct sum decomposition  $V = W \oplus \widetilde{W}$ . This is obviously  $k$ -linear, but it is not obvious whether it preserves the  $G$ -action. The map  $f: V \rightarrow V$  defined as the average

$$(1) \quad f(v) = \frac{1}{|G|} \sum_{s \in G} s(\text{pr}(s^{-1}(v)))$$

is  $k$ -linear, has values in  $W$  since  $\text{pr}$  has values in  $W$  and  $W$  is a  $k[G]$ -submodule of  $V$ , and  $f$  preserves the  $G$ -action:  $f(g(v)) = g(f(v))$  for all  $v \in V$  and  $g \in G$ . Indeed, for  $g \in G$  and  $v \in V$ ,

$$\begin{aligned} f(g(v)) &= \frac{1}{|G|} \sum_{s \in G} s(\text{pr}(s^{-1}(gv))) \\ &= \frac{1}{|G|} \sum_{s \in G} (gs)(\text{pr}((gs)^{-1}(gv))) \\ &= \frac{1}{|G|} \sum_{s \in G} (gs)(\text{pr}(s^{-1}g^{-1}(gv))) \\ &= g \left( \frac{1}{|G|} \sum_{s \in G} s(\text{pr}(s^{-1}(v))) \right) \\ &= g(f(v)). \end{aligned}$$

For  $w \in W$  we have  $s^{-1}(w) \in W$  for all  $s \in G$ , so

$$f(w) = \frac{1}{|G|} \sum_{s \in G} s(\text{pr}(s^{-1}(w))) = \frac{1}{|G|} \sum_{s \in G} s(s^{-1}(w)) = \frac{1}{|G|} \sum_{s \in G} w = \frac{1}{|G|} |G| w = w.$$

Therefore  $f: V \rightarrow W$  has the “projection property”  $f^2 = f$ : for each  $v \in V$ ,  $f(v) \in W$  so  $f(f(v)) = f(v)$ , and thus  $f^2(v) = f(v)$ .

Set  $W' = \ker(f)$ . This subspace of  $V$  is a  $k[G]$ -submodule: if  $f(w') = 0$  then  $f(g(w')) = g(f(w')) = g(0) = 0$ . We will prove  $V = W \oplus W'$ .

First we show  $V = W + W'$ . For  $v \in V$  we can write  $v = f(v) + (v - f(v))$ , with  $f(v) \in W$ . The difference  $v - f(v)$  is in  $W'$  since  $f(v - f(v)) = f(v) - f^2(v) = f(v) - f(v) = 0$ .

Next we show  $W \cap W' = \{0\}$ . If  $w \in W \cap W'$  then  $w \in W \Rightarrow f(w) = w$ , while  $w \in W' \Rightarrow f(w) = 0$ . Thus  $w = 0$ .  $\square$

It is important in the definition of  $f$  in (1) have  $s^{-1}$  on the inside of the formula. The modified function  $F: V \rightarrow V$  defined by

$$F(v) = \frac{1}{|G|} \sum_{s \in G} s(\text{pr}(v))$$

has values in  $W$  but  $F$  need not preserve the  $G$ -action. Here is an example of that.

**Example 2.** Let  $G = \{e, g\}$  have order 2 and let  $G$  act on  $V = k^2$  (where  $2 \neq 0$  in  $k$ ) by having  $e$  act as the identity and  $g$  act as the coordinate swap:  $g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$ . Then  $W = k\begin{pmatrix} 1 \\ 1 \end{pmatrix}$  is a subrepresentation of  $V$  and a complementary subspace to  $W$  is  $\widetilde{W} = k\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ . Then  $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ y \end{pmatrix} + \begin{pmatrix} x-y \\ 0 \end{pmatrix}$  expresses each element of  $V$  according to the decomposition  $W \oplus \widetilde{W}$ , so  $\text{pr}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ y \end{pmatrix}$ . Then

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2} \left( \text{pr}\begin{pmatrix} x \\ y \end{pmatrix} + g\left(\text{pr}\begin{pmatrix} x \\ y \end{pmatrix}\right) \right) = \frac{1}{2} \left( \begin{pmatrix} y \\ y \end{pmatrix} + g\begin{pmatrix} y \\ y \end{pmatrix} \right) = \frac{1}{2} \left( \begin{pmatrix} y \\ y \end{pmatrix} + \begin{pmatrix} y \\ y \end{pmatrix} \right) = \begin{pmatrix} y \\ y \end{pmatrix},$$

so  $F(g\begin{pmatrix} x \\ y \end{pmatrix}) = F\begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} x \\ x \end{pmatrix}$  while  $g\left(F\begin{pmatrix} x \\ y \end{pmatrix}\right) = g\begin{pmatrix} y \\ y \end{pmatrix} = \begin{pmatrix} y \\ y \end{pmatrix}$ . Thus when  $x \neq y$  in  $k$  we have  $F(g\begin{pmatrix} x \\ y \end{pmatrix}) \neq g(F\begin{pmatrix} x \\ y \end{pmatrix})$ .

Unlike  $\ker f$  in the proof of Theorem 1,  $\ker F = k\begin{pmatrix} 1 \\ 0 \end{pmatrix}$  is *not* a subrepresentation of  $k^2$ .

The condition that  $|G| \neq 0$  in  $k$  in Theorem 1 is necessary for the proof to work, since the formula for  $f$  involves division by  $|G|$ . If  $|G| = 0$  in  $k$  then problems really can occur: a subrepresentation need not have a complementary subrepresentation.

**Example 3.** Let  $k$  be a field with prime characteristic  $p$  and  $G = \mathbf{Z}/(p)$ . Then there is a representation of  $G$  on  $V = k^2$  by  $a \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+ay \\ y \end{pmatrix}$ .

The subspace  $W = k\begin{pmatrix} 1 \\ 0 \end{pmatrix}$  is a subrepresentation of  $V$  that has no complementary subrepresentation in  $V$ : if  $W'$  were a complementary subrepresentation of  $W$  then  $\dim(W') = 1$  and each element of  $G$  has  $W'$  as an eigenspace. In particular,  $W'$  is an eigenspace for the matrix  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ . Its only eigenvalue in  $k$  is 1 and its 1-eigenspace in  $V$  is  $W$ , so we have a contradiction.

**Theorem 4.** *Let  $k$  be a field with prime characteristic  $p$ ,  $G$  be a finite group of  $p$ -power order, and  $V$  be a finite-dimensional  $k$ -vector space on which there is a nontrivial representation of  $G$ . Then some subrepresentation of  $V$  does not have a complementary subrepresentation.*

This theorem is about groups with order equal to a power of  $p$ , not groups with the weaker property of having order divisible by  $p$ .

*Proof.* We begin by showing every representation of  $G$  on a nonzero finite-dimensional  $k$ -vector space  $U$  has a one-dimensional trivial subrepresentation. Pick  $u \neq 0$  in  $U$ . Then  $W = \mathbf{F}_p[G]u$  is an  $\mathbf{F}_p$ -vector subspace of  $V$  that is  $G$ -stable and *finite*, so the action of  $G$  on  $W$  is an action of a finite  $p$ -group on a finite set. A theorem about group actions says that for an action of a finite  $p$ -group on a finite set, the number of fixed points is divisible by  $p$ . Since 0 is a fixed point, there must be a nonzero fixed point  $w \in W$ . Then  $g(w) = w$  for all  $g \in G$ , so  $kw$  is a one-dimensional subspace of  $U$  on which  $G$  acts trivially.

Returning to the theorem we want to prove, assume every subrepresentation of  $V$  has a complementary subrepresentation. Then forming nonzero complementary subrepresentations until we can't go any further (this will happen since  $\dim_k(V)$  is finite),  $V$  is a direct sum of subrepresentations  $V_i$  that are irreducible: each one is nonzero and has no proper nonzero subrepresentations. From the first paragraph, each  $V_i$  has a one-dimensional subrepresentation on which  $G$  acts trivially, so by irreducibility  $V_i$  is a one-dimensional trivial representation of  $G$ . Then the direct sum  $V$  is a trivial representation of  $G$ , which is a contradiction of the hypothesis that  $V$  is not a trivial representation for  $G$ .  $\square$