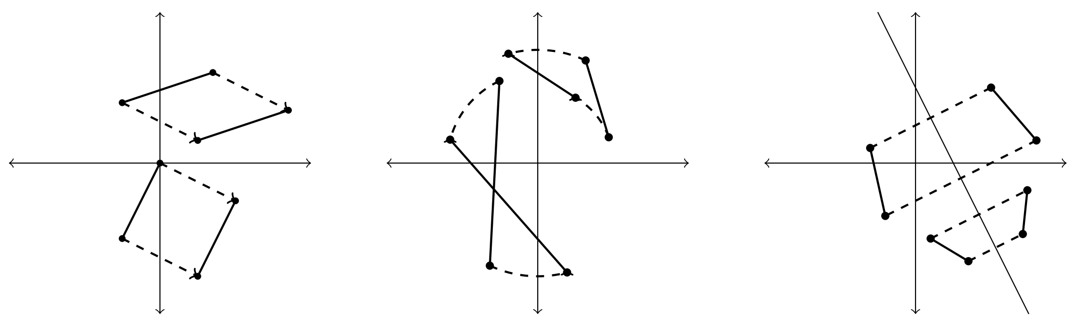
PLANE ISOMETRIES AND THE COMPLEX NUMBERS

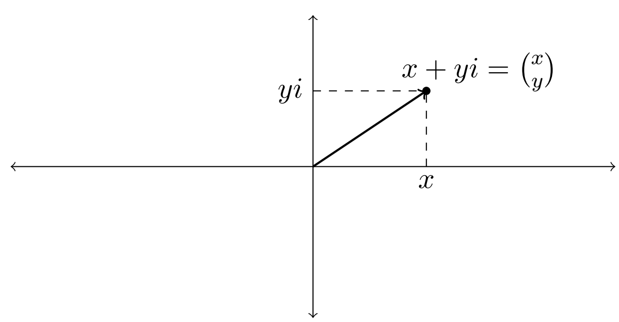
KEITH CONRAD

1. Introduction

The length of a vector $\mathbf{v} = \binom{x}{y}$ in $\mathbf{R}^2$ is $||\binom{x}{y}|| = \sqrt{x^2+y^2}$, and the distance between two vectors $\mathbf{v}$ and $\mathbf{w}$ is the length of their difference: $||\mathbf{v}-\mathbf{w}||$. A function $h\colon \mathbf{R}^2 \to \mathbf{R}^2$ is called an *isometry* when it preserves distances: $||h(\mathbf{v})-h(\mathbf{w})|| = ||\mathbf{v}-\mathbf{w}||$ for all $\mathbf{v}$ and $\mathbf{w}$ in $\mathbf{R}^2$. Examples of isometries include translation by $\binom{1}{-1/2}$, a rotation by 40 degrees counterclockwise around the origin, and reflection across the line $y = 1-2x$. The effect of these isometries on some line segments is illustrated in the figures below.



Isometries of $\mathbf{R}^2$ can be described using linear algebra [1, Chap. 6], and this point of view is essential to discuss isometries in $\mathbf{R}^n$ [2, Sect. 6.5, 6.11]. However, in the special case of the plane we can describe isometries without linear algebra, using instead complex numbers by viewing each vector $\binom{x}{y}$ as the complex number $x+yi$.



Algebraically, addition of vectors and addition of the corresponding complex numbers match, since we add vectors componentwise and add complex numbers by adding the real parts and adding the imaginary parts (which is also componentwise). Geometrically, the length of a vector equals the absolute value of the corresponding complex number: $||\binom{x}{y}|| = \sqrt{x^2+y^2} = |x+yi|$, which is the length of the line segment connecting $x+yi$ to the origin in the figure above.

The distance between two complex numbers z and w is $|z-w|$, so if the vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbf{R}^2$ correspond to the complex numbers z and w, then $||\mathbf{v}-\mathbf{w}|| = |z-w|$. Therefore if we think of $\mathbf{R}^2$ as $\mathbf{C}$, an isometry of $\mathbf{R}^2$ can be viewed as a function $h\colon \mathbf{C} \to \mathbf{C}$ satisfying $|h(z)-h(w)| = |z-w|$ for all z and w in $\mathbf{C}$. Our goal is to describe all the isometries of the plane using algebraic operations with complex numbers. The reason that we will be able to do this is based on four features of complex numbers.

(1) Addition by $x+yi$ is the *translation* of the plane by the vector (x, y).
(2) Multiplication by $\cos\theta + i\sin\theta$, a complex number of absolute value 1, is a counterclockwise *rotation* around the origin by an angle of θ. In particular, for any nonzero $\gamma \in \mathbf{C}$ the pair $\{\gamma, i\gamma\}$ is an orthogonal basis of $\mathbf{C}$ as a real vector space.
(3) Complex conjugation has the effect of *reflection* across the x-axis, so in particular solutions to $\overline{z} = z$ lie on a line (in fact the real line).
(4) The absolute value on $\mathbf{C}$, which is connected to the distance formula via $|z-w|$, behaves nicely with respect to multiplication in $\mathbf{C}$ and complex conjugation:

$$|zw| = |z||w| \text{ and } |\overline{z}| = |z|. \tag{1.1}$$

In Section 2, we will algebraically derive formulas for all isometries of the plane using complex numbers. These formulas will be interpreted geometrically in Section 3.

2. Algebraic arguments

For all $z \in \mathbf{C}$, consider the two formulas

$$h(z) = \alpha z + \beta \text{ or } h(z) = \alpha\overline{z} + \beta, \tag{2.1}$$

where $|\alpha| = 1$ and β is any complex number. These formulas are both isometries. Indeed, subtract two values of each formula:

$$h(z) - h(w) = \alpha(z-w) \text{ or } h(z) - h(w) = \alpha\overline{(z-w)}.$$

By (1.1), $|h(z) - h(w)| = |z-w|$ in either case.

We will show every isometry of the plane is given by one of the formulas in (2.1).

Lemma 2.1. *An isometry of the plane that fixes* 0, 1, *and* i *is the identity.*

Proof. Let h be such an isometry. Since $h(0) = 0$, $h(1) = 1$, and $h(i) = i$, for any $z \in \mathbf{C}$ the formula $|h(z) - h(w)| = |z-w|$ for $w = 0$, 1, and i becomes

$$|h(z)| = |z|, \quad |h(z)-1| = |z-1|, \quad |h(z)-i| = |z-i|. \tag{2.2}$$

This says $h(z)$ and z have the same distances from the three numbers 0, 1, and i. Geometrically, it is plausible that a complex number is completely determined by its distances from 0, 1, and i: three circles centered at each of these points can't intersect in more than one common point. Granting this, we conclude from (2.2) that $h(z) = z$.

To check this algebraically, we want to show for w and z in $\mathbf{C}$ that

$$|w| = |z|,\ |w-1| = |z-1|,\ |w-i| = |z-i| \Longrightarrow w = z. \tag{2.3}$$

Square the three equations in (2.3) and use the formula $|u|^2 = u\overline{u}$ for $u \in \mathbf{C}$:

$$w\overline{w} = z\overline{z}, \quad (w-1)(\overline{w}-1) = (z-1)(\overline{z}-1), \quad (w-i)(\overline{w}+i) = (z-i)(\overline{z}+i).$$

Expanding the second and third equations, and feeding in the first, we obtain after simplifying

$$w + \overline{w} = z + \overline{z}, \quad w - \overline{w} = z - \overline{z}.$$

Therefore w and z have equal real parts and equal imaginary parts, so $w = z$. $\square$

Theorem 2.2. *Any isometry of the plane is given by one of the formulas $h(z) = \alpha z + \beta$ or $h(z) = \alpha\overline{z} + \beta$, where $|\alpha| = 1$.*

Proof. In both formulas, $\beta = h(0)$ and $\alpha = h(1) - h(0)$. This suggests that, given an unknown isometry $h\colon \mathbf{C} \to \mathbf{C}$, we *define*

$$\beta = h(0), \quad \alpha = h(1) - h(0).$$

Then $|\alpha| = |h(1) - h(0)| = |1 - 0| = 1$. Now consider the function

$$k(z) := \frac{h(z) - \beta}{\alpha} = \frac{h(z) - h(0)}{h(1) - h(0)}. \tag{2.4}$$

We *expect* this function is either z or $\overline{z}$.

Note first of all that $k(z)$ is an isometry:

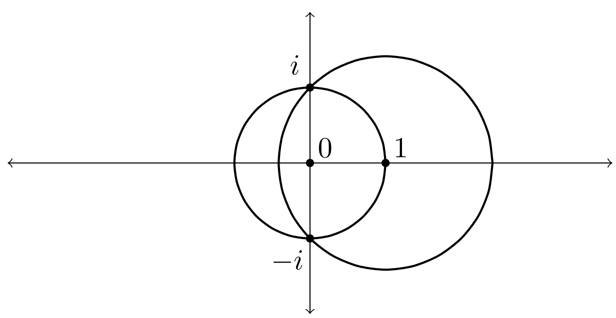
$$|k(z) - k(w)| = \left|\frac{(h(z) - \beta) - (h(w) - \beta)}{\alpha}\right| = |h(z) - h(w)| = |z - w|.$$

With Lemma 2.1 in mind, we compute $k(z)$ at $z = 0, 1$, and i. Easily $k(0) = 0$ and $k(1) = 1$. What about $k(i)$?

Since k fixes 0 and 1,

$$|k(i)| = |k(i) - k(0)| = |i - 0| = 1, \quad |k(i) - 1| = |k(i) - k(1)| = |i - 1| = \sqrt{2},$$

so $k(i)$ lies on both the unit circle and the circle around 1 of radius $\sqrt{2}$. There are only two such points: i and $-i$.



If $k(i) = i$, then k is an isometry fixing 0, 1, and i, so $k(z) = z$ for all z by Lemma 2.1. If $k(i) = -i$, then $\overline{k(z)}$ is an isometry fixing 0, 1, and i, so Lemma 2.1 tells us $\overline{k(z)} = z$ for all z. Conjugating, $k(z) = \overline{z}$ for all z.

Since $h(z) = \alpha k(z) + \beta$, the two possible formulas for $k(z)$ lead to the desired possible formulas for $h(z)$. □

Corollary 2.3. *Any isometry of the plane is an invertible function and its inverse function is an isometry.*

Proof. If $h(z) = \alpha z + \beta$ with $|\alpha| = 1$, its inverse is $(1/\alpha)z - \beta/\alpha$ with $|1/\alpha| = 1$. If $h(z) = \alpha\overline{z} + \beta$ with $|\alpha| = 1$, its inverse is $(1/\overline{\alpha})\overline{z} - \overline{\beta/\alpha}$ with $|1/\overline{\alpha}| = 1$. Both inverses have the shape of the formulas in Theorem 2.2 (*e.g.*, the coefficient of z or $\overline{z}$ has absolute value 1), so these inverse functions are isometries. □

Now we can generalize Lemma 2.1.

Lemma 2.4. *If an isometry of the plane fixes three points that are not collinear, then that isometry is the identity.*

We need to assume in the lemma that the isometry has at least *three* fixed points that are not collinear, since any two points are collinear and reflections fix a line of points.

Proof. We can write an isometry $h(z)$ as $\alpha z + \beta$ or $\alpha\overline{z} + \beta$, where $|\alpha| = 1$. Unless $h(z)$ is the identity, we will show $h(z)$ can't have 3 fixed points that don't lie on a line.

Case 1: $h(z) = \alpha z + \beta$. If $\alpha \neq 1$ then there is only one solution to $h(z) = z$, namely $z = \beta/(1-\alpha)$. If $\alpha = 1$, so $h(z) = z + \beta$, then there are no solutions to $h(z) = z$ if $\beta \neq 0$. So unless $\alpha = 1$ and $\beta = 0$, which means h is the identity, $h(z)$ has at most 1 fixed point.

Case 2: $h(z) = \alpha\overline{z} + \beta$. The condition $h(z) = z$ is the same as $\alpha\overline{z} + \beta = z$. We will show the set of solutions to this equation, if it is not empty, lie along a line. Thus h doesn't have three fixed points not lying on a line. Let w and z both be fixed points of h, so $\alpha\overline{z} + \beta = z$ and $\alpha\overline{w} + \beta = w$ Subtracting one equation from the other,

$$\alpha(\overline{w - z}) = w - z. \tag{2.5}$$

If α were 1, then $\overline{w - z} = w - z$, so $t := w - z$ is real and $w = z + t$. This means all fixed points of h are on the horizontal line through z, so three fixed points of h can't be collinear. What if $\alpha \neq 1$? Here there is a clever trick with square roots. All complex numbers are squares, so we can write $\alpha = \gamma^2$. Then $|\gamma| = 1$, so $\overline{\gamma} = 1/\gamma$, which means $\alpha = \gamma/\overline{\gamma}$ and (2.5) becomes

$$\overline{\left(\frac{w - z}{\gamma}\right)} = \frac{w - z}{\gamma}.$$

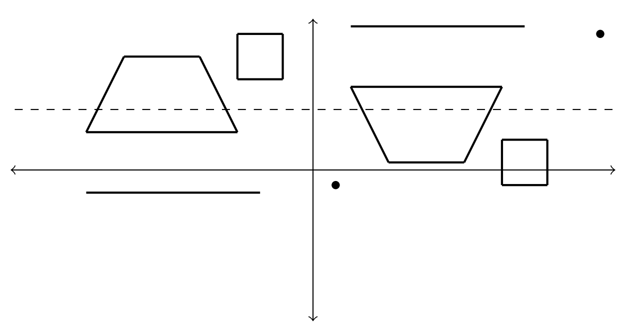
Thus $t := (w - z)/\gamma$ is real, so $w = z + t\gamma$. Now w lies on the line with direction γ passing through z, so again all fixed points of h lie on a line. $\square$

Theorem 2.5. *An isometry of the plane is determined by its values at three noncollinear points: if two isometries h_1 and h_2 are equal at three noncollinear points, then $h_1 = h_2$ everywhere on the plane.*

Proof. The isometry h_2 is invertible and its inverse is an isometry, by Corollary 2.3. To say $h_1(z) = h_2(z)$ is the same as saying $(h_2^{-1}h_1)(z) = z$. Therefore our hypothesis says the composite isometry $h_2^{-1}h_1$ fixes three noncollinear points, so by Lemma 2.4 it is the identity: $(h_2^{-1}h_1)(z) = z$ for all $z \in \mathbf{C}$, so $h_1(z) = h_2(z)$ for all z. $\square$

3. Geometric arguments

Now we want to understand the geometry behind the formulas in (2.1). We will see that the isometries of the plane fall into five types: the identity, translations, rotations, reflections, and glide reflections. A *glide reflection* is a composition of a reflection and a nonzero translation in a direction parallel to the line of reflection. One glide reflection is in the figure below, where the line of reflection is dashed and the translation is to the right.



This image, which includes "before" and "after" states, suggests a physical interpretation of a glide reflection: it is the result of turning the plane in space like a half-turn of a screw.

We will look at the functions in (2.1) and see how they are one of the five types of isometries in terms of conditions on the parameters α and β in each case. Table 1 summarizes the relations we will prove between the geometric description of an isometry and the formula for the isometry. In the table, $|\alpha| = 1$ and $\gamma^2 = \alpha$.

Isometry	Formula	Fixed points
Identity	z	$\mathbf{C}$
Nonzero Translation	$z + \beta,\ \beta \neq 0$	$\emptyset$
Nonzero Rotation	$\alpha z + \beta, \alpha \neq 1$	$\beta/(1-\alpha)$
Reflection	$\alpha\overline{z} + \beta,\ \beta^2/\alpha \leq 0$	$\beta/2 + \mathbf{R}\gamma$
Glide Reflection	$\alpha\overline{z} + \beta,\ \beta^2/\alpha \not\leq 0$	$\emptyset$

TABLE 1. Isometries of $\mathbf{R}^2$

In this table, the only nonidentity isometries fixing more than one point are reflections, whose fixed points lie along a line. This is a better way of understanding Lemma 2.4.

Now we turn to the justification of the information in Table 1.

Case 1: $h(z) = z + \beta$. This is a translation of the plane by β. It has no fixed points unless $\beta = 0$, in which case h is the identity and all points are fixed.

Case 2: $h(z) = \alpha z + \beta$, $\alpha \neq 1$.

The unique fixed point of h is $z_0 = \beta/(1-\alpha)$. We expect h should be a rotation around z_0. To prove this, express $h(z)$ in terms of its fixed point z_0:

$$h(z) = \alpha z + \beta = \alpha z + (1-\alpha)z_0 = \alpha(z - z_0) + z_0. \tag{3.1}$$

Since multiplication by α (of absolute value 1, and not equal to 1) is a nontrivial rotation around 0, (3.1) tells us h is a rotation around z_0: the process of subtracting z_0, multiplying by α, and then adding back z_0 amounts to rotating around z_0 by the nonzero angle determined by α as a rotation around 0.

Case 3: $h(z) = \alpha\overline{z} + \beta$. There may be some fixed points or there may be no fixed points. For instance, the equation $\overline{z} = z$ has the real line as its fixed points, while the equation $\overline{z} + 1 = z$ has no fixed points. It will turn out that h has either a whole line of fixed points (in which case we will show h is the reflection across that line) or it has no fixed points (in which case we will show h is a glide reflection).

In the proof of Lemma 2.4 we studied the equation $\alpha\overline{z} + \beta = z$ using a square root of α. Let's bring back that square root. Write $\alpha = \gamma^2$, so $|\gamma| = 1$ (since $|\alpha| = 1$). From the proof of Lemma 2.4, if w and z are both fixed points of h then

$$\overline{\left(\frac{w-z}{\gamma}\right)} = \frac{w-z}{\gamma},$$

so $t := (w-z)/\gamma$ is real and $w = z + t\gamma$. Conversely, for any real t and complex number z such that $\alpha\overline{z} + \beta = z$, the reader can check that the number $w = z + t\gamma$ satisfies $\alpha\overline{w} + \beta = w$. Therefore the set of fixed points of $h(z) = \alpha\overline{z} + \beta$ is

$$\{z_0 + t\gamma : t \in \mathbf{R}\} = z_0 + \mathbf{R}\gamma, \tag{3.2}$$

where z_0 is any one fixed point.

It need not happen that $\alpha\overline{z}+\beta$ has any fixed points (*e.g.*, $h(z)=\overline{z}+1$). A necessary and sufficient condition on α and β in order for the isometry $h(z)=\alpha\overline{z}+\beta$ to have a fixed point is $\beta^2/\alpha\leq 0$. To see why this is necessary, suppose $\alpha\overline{z}+\beta=z$ for some z. Applying complex conjugation to this equation gives us $\overline{\alpha}z+\overline{\beta}=\overline{z}$, so we must have

$$\overline{z}=\frac{z-\beta}{\alpha}=\overline{\alpha}z+\overline{\beta}.$$

Since $\overline{\alpha}=1/\alpha$, the terms z/α and $\overline{\alpha}z$ in the second equation above match, and subtracting them leaves us with $-\beta/\alpha=\overline{\beta}$, so $\beta^2/\alpha=-\beta\overline{\beta}=-|\beta|^2\leq 0$. Conversely, if $\beta^2/\alpha=-c$ where $c\geq 0$, then $|\beta|^2=c$ and (miracle!) the number $\beta/2$ is a fixed point of h:

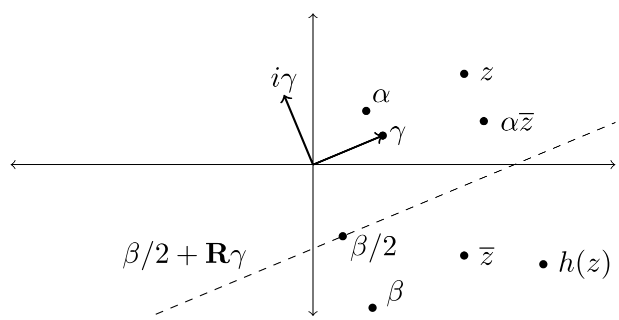
$$h\left(\frac{\beta}{2}\right)=\alpha\cdot\frac{\overline{\beta}}{2}+\beta=-\frac{\beta^2\overline{\beta}}{2c}+\beta=-\frac{\beta|\beta|^2}{2c}+\beta=-\frac{\beta}{2}+\beta=\frac{\beta}{2}.$$

To summarize, $h(z)=\alpha\overline{z}+\beta$ has a fixed point if and only if $\beta^2/\alpha\leq 0$, in which case $\beta/2$ is a fixed point, so (3.2) tells us that the set of all fixed points of h is the line

$$\frac{\beta}{2}+\mathbf{R}\gamma,$$

where $\gamma^2=\alpha$. See the figure below. We will prove h is reflection across the line $\beta/2+\mathbf{R}\gamma$ by checking two properties:

- for all z, the average $\frac{1}{2}(z+h(z))$ lies on the line $\beta/2+\mathbf{R}\gamma$,
- for all z, the difference $z-h(z)$ is perpendicular to the line $\beta/2+\mathbf{R}\gamma$.



For the first property,

$$\begin{aligned}\frac{z+h(z)}{2} &= \frac{z+\alpha\overline{z}+\beta}{2}\\ &= \frac{\beta}{2}+\frac{z+\gamma^2\overline{z}}{2}\\ &= \frac{\beta}{2}+\frac{\overline{\gamma}z+\gamma\overline{z}}{2}\gamma.\end{aligned}$$

The number $\overline{\gamma}z+\gamma\overline{z}=\overline{\gamma}z+\overline{\overline{\gamma}z}$ is real, so this average is on the line $\beta/2+\mathbf{R}\gamma$. For the second property,

$$\begin{aligned}z-h(z) &= z-(\alpha\overline{z}+\beta)\\ &= \gamma\overline{\gamma}z-\gamma^2\overline{z}-\beta\\ &= (\overline{\gamma}z-\gamma\overline{z})\gamma-\beta.\end{aligned}$$

We want this to be orthogonal to the direction of γ (which is the direction of the line $\beta/2+\mathbf{R}\gamma$). Multiplication of a complex number by i rotates it 90 degrees, so the direction orthogonal to γ is the direction of $i\gamma$. The difference $\overline{\gamma}z - \gamma\overline{z} = \overline{\gamma}z - \overline{\overline{\gamma}z}$ is purely imaginary, so the first term in the formula for $z - h(z)$ is in $\mathbf{R}i\gamma$. What about $-\beta$? Since $\beta^2/\alpha \leq 0$, write $\beta^2 = -\alpha c^2$ for real c. Then $\beta^2 = -\gamma^2 c^2$, so $\beta = \pm ic\gamma \in \mathbf{R}i\gamma$. Thus $z - h(z) \in \mathbf{R}i\gamma$. That completes the proof that h is the reflection across the line $\beta/2 + \mathbf{R}\gamma$.

The remaining type of isometry is $h(z) = \alpha\overline{z} + \beta$ with no fixed points, so $\beta^2/\alpha \not\leq 0$. Before we show h is a glide reflection (that is, a composition of a reflection and a nonzero translation in a direction parallel to the line of reflection), let's look at two examples.

Example 3.1. Let $h(z) = \overline{z} + 1$. This is a reflection across the x-axis composed with a translation (by 1) in a direction parallel to the line of reflection.

Example 3.2. Let $h(z) = \overline{z} + (1 + i) = (\overline{z} + i) + 1$. Since $i = w - \overline{w}$ for $w = i/2$, the function $z \mapsto \overline{z} + i$ is a reflection across the horizontal line $i/2 + \mathbf{R}$. Thus h is a composition of a reflection across a line and a translation (by 1) in a direction parallel to that line.

To see that $h(z) = \alpha\overline{z} + \beta$ describes a glide reflection when it has no fixed points, write $\alpha = \gamma^2$. Using $\{\gamma, i\gamma\}$ as an $\mathbf{R}$-basis of $\mathbf{C}$, we can write $\beta = a\gamma + bi\gamma$ with real a and b. Then

$$h(z) = \alpha\overline{z} + \beta = \alpha\overline{z} + (a\gamma + bi\gamma) = (\alpha\overline{z} + bi\gamma) + a\gamma. \tag{3.3}$$

Since $(bi\gamma)^2/\alpha = -b^2 \leq 0$, our earlier discussion of reflections tells us that $\alpha\overline{z} + bi\gamma$ is a reflection across a line parallel to $\mathbf{R}\gamma$. Since $\beta^2/\alpha \not\leq 0$, we must have $a \neq 0$. Therefore adding $a\gamma$ in (3.3) amounts to composition of a reflection across a line parallel to $\mathbf{R}\gamma$ with a nonzero translation in a direction parallel to the same line. Hence h is a glide reflection.

This completes the justification of Table 1, although we should also emphasize why the different types of isometries in Table 1 are different from each other. Looking at the set of fixed points (empty, point, line, plane) easily distinguishes the first four types of isometries from each other and distinguishes a glide reflection from all but a translation. To show a glide reflection is not a translation, if $h(z) = \alpha\overline{z} + \beta$ were a translation then $h(z) - z$ would be constant, but $h(z) - z = \alpha\overline{z} - z + \beta$ is not constant: from $h(0) - 0 = h(1) - 1 = h(i) - i$ the first equation would imply $\alpha = 1$ and the second would imply $\alpha = -i$.

We now draw a few conclusions about plane isometries from the formulas in Table 1.

Theorem 3.3. *The composition of two reflections is a translation or a rotation, and it is a translation precisely when the two reflections are across a pair of parallel lines.*

Proof. Let s_1 and s_2 be two reflections, so $s_1(z) = \alpha_1\overline{z} + \beta_1$ and $s_2(z) = \alpha_2\overline{z} + \beta_2$, where α_1 and α_2 have absolute value 1 and β_1^2/α_1 and β_2^2/α_2 are ≤ 0. Then

$$s_1(s_2(z)) = \alpha_1\overline{(\alpha_2\overline{z} + \beta_2)} + \beta_1 = (\alpha_1\overline{\alpha}_2)z + (\alpha_1\overline{\beta}_2 + \beta_1).$$

This has the form $\alpha z + \beta$ where $\alpha = \alpha_1\overline{\alpha}_2$, so $|\alpha| = 1$. Table 1 tells us that $s_1 \circ s_2$ is a translation or rotation (possibly the identity).

The composite $s_1 \circ s_2$ is a translation precisely when $\alpha_1\overline{\alpha}_2 = 1$, which (since $\overline{\alpha}_2 = 1/\alpha_2$) amounts to saying $\alpha_2 = \alpha_1$. Then $s_1(z) = \alpha_1 z + \beta_1$ and $s_2(z) = \alpha_1 z + \beta_2$. The fixed lines of these reflections are $\beta_1/2 + \mathbf{R}\gamma$ and $\beta_2/2 + \mathbf{R}\gamma$, where $\gamma^2 = \alpha_1$, so these lines are parallel. $\square$

Example 3.4. Let $s_1(z) = -\overline{z}+3$, $s_2(z) = -\overline{z}+5$, and $s_3(z) = \overline{z}+i$. Their respective lines of reflection are $3/2 + \mathbf{R}i$ (horizontal), $5/2 + \mathbf{R}i$ (horizontal), and $i/2 + \mathbf{R}$ (vertical). The composite $s_1(s_2(z))$ is $z - 2$, which is a translation. The composite $s_1(s_3(z))$ is $-z + (3 + i)$, which is a rotation around $(3 + i)/2$.

Theorem 3.5. *If g and h are two isometries of the plane, ghg^{-1} is an isometry of the same type as h.*

Proof. Write $g(z) = \alpha z + \beta$. For a translation $t(z) = z + c$, an explicit calculation shows $(gtg^{-1})(z) = z + \alpha c$. If we have $\overline{z}$ in place of z in $g(z)$, then $(gtg^{-1})(z) = z + \alpha\overline{c}$. Thus gtg^{-1} is a translation for any isometry g.

For any two plane isometries g and h, h fixes z if and only if ghg^{-1} fixes $g(z)$. Any isometry takes a point or a line to a point or a line, respectively, so from our knowledge of fixed points of different kinds of isometries we see that if h is a rotation then ghg^{-1} is a rotation (1 fixed point), and if h is a reflection then ghg^{-1} is a reflection (a line of fixed points).

(We can say something more precise in the case $h(z) = \alpha z + \beta$ is a rotation. When g is a translation or rotation $(ghg^{-1})(z) = \alpha z + *$, while when g is a reflection or glide reflection $(ghg^{-1})(z) = \overline{\alpha} z + *$. Therefore ghg^{-1} is a rotation by the same angle as h or the opposite angle to h, at some point in the plane.)

Suppose h is a glide reflection, so it has no fixed points. Then ghg^{-1} also has no fixed points, which means ghg^{-1} is either a glide reflection or a translation. If ghg^{-1} were a translation, say $ghg^{-1} = t$, then $h = g^{-1}tg$, which is a translation by the first paragraph (replace g with g^{-1} there). That is a contradiction, so ghg^{-1} must be a glide reflection. □

Theorem 3.6. *Any isometry of $\mathbf{R}^2$ is a composition of at most* 2 *reflections, except for glide reflections, which are compositions of* 3 (*and no fewer*) *reflections.*

Proof. We check the result for the identity, nonzero translations, rotations, and glide reflections.

The identity is the composition of any reflection with itself. (For instance, $z = \overline{\overline{z}}$).

Write a nonzero translation as $t(z) = z + \beta$. Set $s(z) = -(\beta/\overline{\beta})\overline{z}$ and $s'(z) = -(\beta/\overline{\beta})\overline{z} + \beta$. Both are reflections by Table 1. (For instance, writing $s'(z) = \alpha\overline{z} + \beta$ we have $\beta^2/\alpha = -\beta\overline{\beta} = -|\beta|^2 \leq 0$). Check by a computation that $(s' \circ s)(z) = z + \beta = t(z)$. (Alternatively, from $s' = t \circ s$ we get $t = s' \circ s^{-1} = s' \circ s$.)

A rotation around 0 can be written as $h(z) = \alpha z$. Take $k(z) = \overline{z}$ and $\ell(z) = \overline{\alpha} \cdot \overline{z}$. Then k and ℓ are reflections by Table 1 and $h = k \circ \ell$. An arbitrary rotation of the plane has the form $h(z) = \alpha(z - z_0) + z_0$, where z_0 is the center of rotation and $|\alpha| = 1$. Then $h = t \circ r \circ t^{-1}$, where t is translation by z_0 and $r(z) = \alpha z$ is a rotation around 0. From what we already said about rotations around 0, we can write $r = k \circ \ell$ for two reflections k and ℓ. Then $h = trt^{-1} = t(k\ell)t^{-1} = (tkt^{-1}) \circ (t\ell t^{-1})$, which is a composition of two reflections by Theorem 3.5.

Lastly, a glide reflection is a composition of a reflection and a nonzero translation. Since a (nonzero) translation is a composition of two reflections, any glide reflection is a composition of 3 reflections. The number of reflections can't be reduced. Indeed, a glide reflection is not a reflection since they don't have the same kind of fixed points (the empty set and a line), and a glide reflection is not a composition of two reflections since the only such isometries are translations and rotations, by Theorem 3.3. □

References

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