GROUPS OF ORDER $p^3$

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For any prime $p$, we want to describe the groups of order $p^3$ up to isomorphism. From the cyclic decomposition of finite abelian groups, there are three abelian groups of order $p^3$ up to isomorphism: $\mathbb{Z}/(p^3)$, $\mathbb{Z}/(p^2) \times \mathbb{Z}/(p)$, and $\mathbb{Z}/(p) \times \mathbb{Z}/(p) \times \mathbb{Z}/(p)$. These are nonisomorphic since they have different maximal orders for their elements: $p^3$, $p^2$, and $p$ respectively. We will show there are two nonabelian groups of order $p^3$ up to isomorphism. The descriptions of these two groups will be different for $p = 2$ and $p \neq 2$, so we will treat these cases separately after the following lemma.

**Lemma 1.** Let $p$ be prime and $G$ be a nonabelian group of order $p^3$ with center $Z$. Then $\#Z = p$, $G/Z \cong (\mathbb{Z}/(p)) \times (\mathbb{Z}/(p))$, and $[G, G] = Z$.

Proof. Since $G$ is a nontrivial group of $p$-power order, its center is nontrivial. Therefore $\#Z = p, p^2$, or $p^3$. Since $G$ is nonabelian, $\#Z \neq p^3$. For any group $G$, if $G/Z$ is cyclic then $G$ is abelian. So $G$ being nonabelian forces $G/Z$ to be noncyclic. Therefore $\#(G/Z) \neq p$, so $\#Z \neq p^2$. The only choice left is $\#Z = p$, so $G/Z$ has order $p^2$.

Up to isomorphism the only groups of order $p^2$ are $\mathbb{Z}/(p^2)$ and $\mathbb{Z}/(p) \times \mathbb{Z}/(p)$. Since $G/Z$ is noncyclic, $G/Z \cong \mathbb{Z}/(p) \times \mathbb{Z}/(p)$.

Since $G/Z$ is abelian, we have $[G, G] \subseteq Z$. Because $\#Z = p$ and $[G, G]$ is nontrivial, necessarily $[G, G] = Z$. \hfill $\square$

**Theorem 2.** A nonabelian group of order 8 is isomorphic to $D_4$ or to $Q_8$.

The groups $D_4$ and $Q_8$ are not isomorphic since there are 5 elements of order 2 in $D_4$ and only one element of order 2 in $Q_8$.

Proof. Let $G$ be nonabelian of order 8. The nonidentity elements in $G$ have order 2 or 4. If $g^2 = 1$ for all $g \in G$ then $G$ is abelian, so some $x \in G$ must have order 4.

Let $y \in G - \langle x \rangle$. The subgroup $\langle x, y \rangle$ properly contains $\langle x \rangle$, so $\langle x, y \rangle = G$. Since $G$ is nonabelian, $x$ and $y$ do not commute.

Since $\langle x \rangle$ has index 2 in $G$, it is a normal subgroup. Therefore $yxy^{-1} \in \langle x \rangle$:

$$yxy^{-1} \in \{1, x, x^2, x^3\}.$$  

Since $yxy^{-1}$ has order 4, $yxy^{-1} = x$ or $yxy^{-1} = x^3 = x^{-1}$. The first option is not possible, since it says $x$ and $y$ commute, which they don’t. Therefore

$$yxy^{-1} = x^{-1}.$$  

The group $G/\langle x \rangle$ has order 2, so $y^2 \in \langle x \rangle$:

$$y^2 \in \{1, x, x^2, x^3\}.$$  

Since $y$ has order 2 or 4, $y^2$ has order 1 or 2. Thus $y^2 = 1$ or $y^2 = x^2$.

Putting this together, $G = \langle x, y \rangle$ where either

$$x^4 = 1, \quad y^2 = 1, \quad yxy^{-1} = x^{-1}$$
or
\[ x^4 = 1, \quad y^2 = x^2, \quad yxy^{-1} = x^{-1}. \]
In the first case \( G \cong D_4 \) and in the second case \( G \cong Q_8 \). \qed

From now on we take \( p \neq 2 \). The two nonabelian groups of order \( p^3 \), up to isomorphism, will turn out to be
\[
\text{Heis}(\mathbb{Z}/(p)) = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{Z}/(p) \right\}
\]
and
\[
G_p = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a, b \in \mathbb{Z}/(p^2), a \equiv 1 \mod p \right\} = \left\{ \begin{pmatrix} 1 + pm & b \\ 0 & 1 \end{pmatrix} : m, b \in \mathbb{Z}/(p^2) \right\},
\]
where \( m \) actually only matters modulo \( p \).\(^1\) These two constructions both make sense at the prime 2, but in that case the two groups are isomorphic to each other, as we’ll see below.

We can distinguish \( \text{Heis}(\mathbb{Z}/(p)) \) from \( G_p \) for \( p \neq 2 \) by counting elements of order \( p \). In \( \text{Heis}(\mathbb{Z}/(p)) \),
\[
\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & na & nb + \frac{n(n-1)}{2}ac \\ 0 & 1 & nc \\ 0 & 0 & 1 \end{pmatrix}
\]
for any \( n \in \mathbb{Z} \), so
\[
\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}^p = \begin{pmatrix} 1 & 0 & p\frac{(p-1)}{2}ac \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]
When \( p \neq 2 \), \( p\frac{(p-1)}{2} \equiv 0 \mod p \), so all nonidentity elements of \( \text{Heis}(\mathbb{Z}/(p)) \) have order \( p \). On the other hand, \( \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \in G_p \) has order \( p^2 \) since \( \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \). So \( G_p \not\cong \text{Heis}(\mathbb{Z}/(p)) \).

At the prime 2, \( \text{Heis}(\mathbb{Z}/(2)) \) and \( G_2 \) each contain more than one element of order 2, so \( \text{Heis}(\mathbb{Z}/(2)) \) and \( G_2 \) are both isomorphic to \( D_4 \).

Let’s look at how matrices combine and decompose in \( \text{Heis}(\mathbb{Z}/(p)) \) and \( G_p \) when \( p \neq 2 \), since this will inform some of our computations later in an abstract nonabelian group of order \( p^3 \). In \( \text{Heis}(\mathbb{Z}/(p)) \),
\[
\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a' & b' \\ 0 & 1 & c' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a + a' & b + b' + ac' \\ 0 & 1 & c + c' \\ 0 & 0 & 1 \end{pmatrix}
\]
and in \( G_p \)
\[
\begin{pmatrix} 1 + pm & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 + pm' & b' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 + p(m + m') & b + b' + pmb' \\ 0 & 1 \end{pmatrix}.
\]

\(^1\)The notation \( G_p \) for this group is not standard. I don’t know a standard notation for it.
In Heis($\mathbb{Z}/(p)$),
\[
\begin{pmatrix}
1 & a & b \\
0 & 1 & c \\
0 & 0 & 1 \\
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & c \\
0 & 0 & 1 \\
\end{pmatrix} \begin{pmatrix}
1 & a & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix} \begin{pmatrix}
1 & 0 & b \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1 \\
\end{pmatrix} c \begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix} a \begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix} b
\]
by (1)

and a particular commutator is
\[
\begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}, \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1 \\
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}.
\]

So if we set
\[
x = \begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}
\]
and
\[
y = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1 \\
\end{pmatrix}
\]

then
\[
\begin{pmatrix}
1 & a & b \\
0 & 1 & c \\
0 & 0 & 1 \\
\end{pmatrix} = y^c x^a [x, y]^b.
\]

In $G_p \subset \text{Aff}(\mathbb{Z}/(p^2))$,
\[
\begin{pmatrix}
1 + pm & b \\
0 & 1 \\
\end{pmatrix} = \begin{pmatrix}
1 & b \\
0 & 1 \\
\end{pmatrix} \begin{pmatrix}
1 + pm & 0 \\
0 & 1 \\
\end{pmatrix} = \begin{pmatrix}
1 & 1 \\
0 & 1 \\
\end{pmatrix} b \begin{pmatrix}
1 + p & 0 \\
0 & 1 \\
\end{pmatrix}^m.
\]

If we set
\[
x = \begin{pmatrix}
1 + p & 0 \\
0 & 1 \\
\end{pmatrix}
\]
and
\[
y = \begin{pmatrix}
1 & 1 \\
0 & 1 \\
\end{pmatrix}
\]

then
\[
\begin{pmatrix}
1 + pm & b \\
0 & 1 \\
\end{pmatrix} = y^b x^m
\]

and
\[
[x, y] = \begin{pmatrix}
1 & p \\
0 & 1 \\
\end{pmatrix} = y^p.
\]

**Lemma 3.** In a group $G$, if $g$ and $h$ commute with $[g, h]$ then $[g^m, h^n] = [g, h]^{mn}$ for all $m$ and $n$ in $\mathbb{Z}$, and $g^n h^n = (gh)^n [g, h]^{\binom{n}{2}}$.

**Proof.** Exercise. □

**Theorem 4.** For primes $p \neq 2$, a nonabelian group of order $p^3$ is isomorphic to Heis($\mathbb{Z}/(p)$) or $G_p$.

**Proof.** Let $G$ be a nonabelian group of order $p^3$. Any $g \neq 1$ in $G$ has order $p$ or $p^2$.

By Lemma 1, we can write $G/Z = \langle \pi, \eta \rangle$ and $Z = \langle z \rangle$. For any $g \in G$, $g \equiv x^i y^j$ mod $Z$ for some integers $i$ and $j$, so $g = x^i y^j z^k = z^k x^i y^j$ for some $k \in \mathbb{Z}$. If $x$ and $y$ commute then $G$ is abelian (since $z^k$ commutes with $x$ and $y$), which is a contradiction. Thus $x$ and $y$ do not commute. Therefore $[x, y] = x y x^{-1} y^{-1} \in Z$ is nontrivial, so $Z = \langle [x, y] \rangle$. Therefore we can use $[x, y]$ for $z$, showing $G = \langle x, y \rangle$.  

Let’s see what the product of two elements of $G$ looks like. Using Lemma 3,

\[(5)\quad x^i y^j = y^j x^i [x, y]^{ij}, \quad y^j x^i = x^i y^j [x, y]^{-ij}.
\]

This shows we can move any power of $y$ past any power of $x$ on either side, at the cost of introducing a (commuting) power of $[x, y]$. So every element of $G = \langle x, y \rangle$ has the form $y^j x^i [x, y]^k$. (We write in this order because of (4).) A product of two such terms is

\[
y^c x^a [x, y]^b \cdot y^c' x^a' [x, y]^{b'} = y^c (x^a y^c') x^a [x, y]^{b+b'} = y^{c+c'} x^{a+a'} [x, y]^{b+b'+ac'}.
\]

Here the exponents are all integers. Comparing this with (2), it appears we have a homomorphism $\text{Heis}(\mathbb{Z}/(p)) \to G$ by

\[(6)\quad \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mapsto y^c x^a [x, y]^b.
\]

After all, we just showed multiplication of such triples $y^c x^a [x, y]^b$ behaves like multiplication in $\text{Heis}(\mathbb{Z}/(p))$. But there is a catch: the matrix entries $a$, $b$, and $c$ in $\text{Heis}(\mathbb{Z}/(p))$ are integers modulo $p$, so the “function” (6) from $\text{Heis}(\mathbb{Z}/(p))$ to $G$ is only well-defined if $x$, $y$, and $[x, y]$ all have $p$-th power 1 (so exponents on them only matter mod $p$). Since $[x, y]$ is in the center of $G$, a subgroup of order $p$, its exponents only matter modulo $p$. But maybe $x$ or $y$ could have order $p^2$.

Well, if $x$ and $y$ both have order $p$, then there is no problem with (6). It is a well-defined function $\text{Heis}(\mathbb{Z}/(p)) \to G$ that is a homomorphism. Since its image contains $x$ and $y$, the image contains $\langle x, y \rangle = G$, so the function is onto. Both $\text{Heis}(\mathbb{Z}/(p))$ and $G$ have order $p^3$, so our surjective homomorphism is an isomorphism: $G \cong \text{Heis}(\mathbb{Z}/(p))$.

What happens if $x$ or $y$ has order $p^2$? In this case we anticipate that $G \cong G_p$. In $G_p$, two generators are $g = \left(\begin{smallmatrix} 1+p & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $h = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, where $g$ has order $p$, $h$ has order $p^2$, and $[g, h] = h^p$. We want to show our abstract $G$ also has a pair of generators like this.

Starting with $G = \langle x, y \rangle$ where $x$ or $y$ has order $p^2$, without loss of generality let $y$ have order $p^2$. It may or may not be the case that $x$ has order $p$. To show we can change generators to make $x$ have order $p$, we will look at the $p$-th power function on $G$. For any $g \in G$, $g^p \in Z$ since $G/Z \cong \mathbb{Z}/(p) \times \mathbb{Z}/(p)$. Moreover, the $p$-th power function on $G$ is a homomorphism: by Lemma 3, $(gh)^p = g^p h^p$ and $[g, h]^p = 1$ since $[G, G] = Z$ has order $p$, so

\[(gh)^p = g^p h^p.
\]

Since $y^p$ has order $p$ and $y^p \in Z$, $Z = \langle y^p \rangle$. Therefore $x^p = (y^p)^r$ for some $r \in \mathbb{Z}$, and since the $p$-th power function on $G$ is a homomorphism we get $(xy^{-r})^p = 1$, with $xy^{-r} \neq 1$ since $x \not\in (y)$. So $xy^{-r}$ has order $p$ and $G = \langle x, y \rangle = \langle xy^{-r}, y \rangle$. We now rename $xy^{-r}$ as $x$, so $G = \langle x, y \rangle$ where $x$ has order $p$ and $y$ has order $p^2$.

We are not guaranteed that $[x, y] = y^p$, which is one of the relations for the two generators of $G_p$. How can we force this relation to occur? Well, since $[x, y]$ is a nontrivial element of $[G, G] = Z$, $Z = \langle [x, y] \rangle = \langle y^p \rangle$, so

\[(7)\quad [x, y] = (y^p)^k,
\]
where \( k \not\equiv 0 \mod p \). Let \( \ell \) be a multiplicative inverse for \( k \mod p \) and raise both sides of (7) to the \( \ell \)th power: using Lemma 3,

\[ [x, y]^\ell = (y^{pk})^\ell \implies [x^\ell, y] = y^p. \]

Since \( \ell \not\equiv 0 \mod p \), \( \langle x \rangle = \langle x^\ell \rangle \), so we can rename \( x^\ell \) as \( x \); now \( G = \langle x, y \rangle \) where \( x \) has order \( p \), \( y \) has order \( p^2 \), and \( [x, y] = y^p \).

Because \( [x, y] \) commutes with \( x \) and \( y \) and \( G = \langle x, y \rangle \), every element of \( G \) has the form \( y^j x^i \). Let’s see how such products multiply:

\[
y^b x^m \cdot y^{b'} x^{m'} = y^{b + b'} x^m (y^{p^{b'b'}}) x^{m'} = y^{b + b' + pmb'} x^{m + m'}.
\]

Comparing this with (3), we have a homomorphism \( G_p \to G \) by

\[
\begin{pmatrix} 1 + pm & b \\ 0 & 1 \end{pmatrix} \mapsto y^b x^m.
\]

(This function is well-defined since on the left side \( m \) matters \( \mod p \) and \( b \) matters \( \mod p^2 \) while \( x^p = 1 \) and \( y^{p^2} = 1 \).) This homomorphism is onto since \( x \) and \( y \) are in the image, so it is an isomorphism since \( G_p \) and \( G \) have equal order: \( G \cong G_p \). \( \square \)

Let’s summarize what can be said about groups of small \( p \)-power order.

- There is one group of order \( p \) up to isomorphism.
- There are two groups of order \( p^2 \) up to isomorphism: \( \mathbb{Z}/(p^2) \) and \( \mathbb{Z}/(p) \times \mathbb{Z}/(p) \).
- There are five groups of order \( p^3 \) up to isomorphism, but our explicit description of them is not uniform in \( p \) since the case \( p = 2 \) used a separate treatment.

For groups of order \( p^4 \), the count is no longer uniform in \( p \): there are 14 groups of order 16 and 15 groups of order \( p^4 \) for \( p \neq 2 \). This was first determined by Hölder (1893), who also classified the groups of order \( p^3 \).