GROUPS OF ORDER 12

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We will use semidirect products to describe all groups of order 12. There turn out to be 5 such groups: 2 are abelian and 3 are nonabelian. The nonabelian groups are an alternating group, a dihedral group, and a third less familiar group.

Theorem 1. *Any group of order* 12 *is isomorphic to one of* $\mathbf{Z}/(12)$, $(\mathbf{Z}/(2))^2 \times \mathbf{Z}/(3)$, A_4, D_6, *or the nontrivial semidirect product* $\mathbf{Z}/(3) \rtimes \mathbf{Z}/(4)$.

In the proof, we will appeal to an isomorphism property of semidirect products: for any semidirect product $H \rtimes_\varphi K$ and automorphism $f\colon K \to K$, $H \rtimes_\varphi K \cong H \rtimes_{\varphi\circ f} K$. This says that precomposing an action of K on H by automorphisms (that's φ) with an automorphism of K produces an isomorphic semidirect product of H and K.

Proof. Let $|G| = 12 = 2^2 \cdot 3$. A 2-Sylow subgroup has order 4 and a 3-Sylow subgroup has order 3. We will start by showing G has a normal 2-Sylow or 3-Sylow subgroup: $n_2 = 1$ or $n_3 = 1$. From the Sylow theorems,

$$n_2 \mid 3, \quad n_2 \equiv 1 \bmod 2, \quad n_3 \mid 4, \quad n_3 \equiv 1 \bmod 3.$$

Therefore $n_2 = 1$ or 3 and $n_3 = 1$ or 4.

To show $n_2 = 1$ or $n_3 = 1$, assume $n_3 \neq 1$. Then $n_3 = 4$. Let's count elements of order 3. Since each 3-Sylow subgroup has prime order 3, two different 3-Sylow subgroups intersect trivially. Each of the four 3-Sylow subgroups of G contains two elements of order 3 and these are not in any other 3-Sylow subgroup, so the number of elements in G of order 3 is $2n_2 = 8$. This leaves us with $12 - 8 = 4$ elements in G not of order 3. A 2-Sylow subgroup has order 4 and contains no elements of order 3, so one 2-Sylow subgroup must account for the remaining 4 elements of G. Thus $n_2 = 1$ if $n_3 \neq 1$.

Next we show G is a semidirect product of a 2-Sylow and 3-Sylow subgroup. Let P be a 2-Sylow subgroup and Q be a 3-Sylow subgroup of G. Since P and Q have relatively prime orders, $P \cap Q = \{1\}$ and the set $PQ = \{xy : x \in P, y \in Q\}$ has size $|P||Q|/|P \cap Q| = 12 = |G|$, so $G = PQ$. Since P or Q is normal in G, G is a semidirect product of P and Q: $G \cong P \rtimes Q$ if $P \lhd G$ and $G \cong Q \rtimes P$ if $Q \lhd G$.[1] Groups of order 4 are isomorphic to $\mathbf{Z}/(4)$ or $(\mathbf{Z}/(2))^2$, and groups of order 3 are isomorphic to $\mathbf{Z}/(3)$, so G is a semidirect product of the form

$$\mathbf{Z}/(4) \rtimes \mathbf{Z}/(3), \quad (\mathbf{Z}/(2))^2 \rtimes \mathbf{Z}/(3), \quad \mathbf{Z}/(3) \rtimes \mathbf{Z}/(4), \quad \mathbf{Z}/(3) \rtimes (\mathbf{Z}/(2))^2.$$

We will determine all these semidirect products, up to isomorphism, by working out all the ways $\mathbf{Z}/(4)$ and $(\mathbf{Z}/(2))^2$ can act by automorphisms on $\mathbf{Z}/(3)$ and all the ways $\mathbf{Z}/(3)$ can act by automorphisms on $\mathbf{Z}/(4)$ and $(\mathbf{Z}/(2))^2$.

First we list the automorphisms of the Sylow subgroups: $\mathrm{Aut}(\mathbf{Z}/(4)) \cong (\mathbf{Z}/(4))^\times = \{\pm 1 \bmod 4\}$, $\mathrm{Aut}((\mathbf{Z}/(2))^2) \cong \mathrm{GL}_2(\mathbf{Z}/(2))$, and $\mathrm{Aut}(\mathbf{Z}/(3)) \cong (\mathbf{Z}/(3))^\times = \{\pm 1 \bmod 3\}$.

[1] The notation $P \rtimes Q$ could refer to more than one group since there could be different actions $Q \to \mathrm{Aut}(P)$ leading to nonisomorphic semidirect products.

Case 1: $n_2 = 1$, $P \cong \mathbf{Z}/(4)$.

The 2-Sylow subgroup is normal, so the 3-Sylow subgroup acts on it. Our group is a semidirect product $\mathbf{Z}/(4) \rtimes \mathbf{Z}/(3)$, for which the action of the second group on the first is through a homomorphism $\varphi\colon \mathbf{Z}/(3) \to (\mathbf{Z}/(4))^\times$. The domain has order 3 and the target has order 2, so this homomorphism is trivial, and thus the semidirect product must be trivial: it's the direct product

$$\mathbf{Z}/(4) \times \mathbf{Z}/(3),$$

which is cyclic of order 12 (generator $(1,1)$).

Case 2: $n_2 = 1$, $P \cong \mathbf{Z}/(2) \times \mathbf{Z}/(2)$.

We need to understand all homomorphisms $\varphi\colon \mathbf{Z}/(3) \to \mathrm{GL}_2(\mathbf{Z}/(2))$. The trivial homomorphism leads to the direct product

$$(\mathbf{Z}/(2))^2 \times \mathbf{Z}/(3).$$

What about nontrivial homomorphisms $\varphi\colon \mathbf{Z}/(3) \to \mathrm{GL}_2(\mathbf{Z}/(2))$? Inside $\mathrm{GL}_2(\mathbf{Z}/(2))$, which has order 6 (it's isomorphic to S_3), there is one subgroup of order 3: $\{(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}), (\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}), (\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix})\}$. A nontrivial homomorphism $\varphi\colon \mathbf{Z}/(3) \to \mathrm{GL}_2(\mathbf{Z}/(2))$ is determined by where it sends $1 \bmod 3$, which must go to a solution of $A^3 = I_2$; then $\varphi(k \bmod 3) = A^k$ in general. For φ to be nontrivial, A needs to have order 3, and there are two choices for that. The two matrices of order 3 in $\mathrm{GL}_2(\mathbf{Z}/(2))$ are inverses. Call one of them A, making the other A^{-1}. The resulting homomorphisms $\mathbf{Z}/(3) \to \mathrm{GL}_2(\mathbf{Z}/(2))$ are $\varphi(k \bmod 3) = A^k$ and $\psi(k \bmod 3) = A^{-k}$, which are related to each other by composition with inversion, but *watch out*: inversion is not an automorphism of $\mathrm{GL}_2(\mathbf{Z}/(2))$. It is an automorphism of $\mathbf{Z}/(3)$, where it's negation. So precomposing φ with negation on $\mathbf{Z}/(3)$ turns φ into ψ: $\psi = \varphi \circ f$, where $f(x) = -x$ on $\mathbf{Z}/(3)$. Therefore the two nontrivial homomorphisms $\mathbf{Z}/(3) \to \mathrm{GL}_2(\mathbf{Z}/(2))$ are linked through precomposition with an automorphism of $\mathbf{Z}/(3)$, and therefore φ and ψ define isomorphic semidirect products. This means that up to isomorphism, there is one nontrivial semidirect product

$$(\mathbf{Z}/(2))^2 \rtimes \mathbf{Z}/(3).$$

That is, we have shown that up to isomorphism there is only one group of order 12 with $n_2 = 1$ and 2-Sylow subgroup isomorphic to $\mathbf{Z}/(2) \times \mathbf{Z}/(2)$. The group A_4 fits this description: its normal 2-Sylow subgroup is $\{(1), (12)(34), (13)(24), (14)(23)\}$, which is not cyclic.

Now assume $n_2 \neq 1$, so $n_2 = 3$ and $n_3 = 1$. Since $n_2 > 1$, the group is nonabelian, so it's a nontrivial semidirect product (a direct product of abelian groups is abelian).

Case 3: $n_2 = 3$, $n_3 = 1$, and $P \cong \mathbf{Z}/(4)$.

Our group looks like $\mathbf{Z}/(3) \rtimes \mathbf{Z}/(4)$, built from a nontrivial homomorphism $\varphi\colon \mathbf{Z}/(4) \to \mathrm{Aut}(\mathbf{Z}/(3)) = (\mathbf{Z}/(3))^\times$ There is only one choice of φ: it has to send $1 \bmod 4$ to $-1 \bmod 3$, which determines everything else: $\varphi(c \bmod 4) = (-1)^c \bmod 3$. Therefore there is one nontrivial semidirect product $\mathbf{Z}/(3) \rtimes \mathbf{Z}/(4)$. Explicitly, this group is the set $\mathbf{Z}/(3) \times \mathbf{Z}/(4)$ with group law

$$(a,b)(c,d) = (a + (-1)^b c, b + d).$$

Case 4: $n_2 = 3$, $n_3 = 1$, and $P \cong \mathbf{Z}/(2) \times \mathbf{Z}/(2)$.

The group is $\mathbf{Z}/(3) \rtimes (\mathbf{Z}/(2))^2$ for a nontrivial homomorphism $\varphi\colon (\mathbf{Z}/(2))^2 \to (\mathbf{Z}/(3))^\times$. The group $(\mathbf{Z}/(2))^2$ has a pair of generators $(1,0)$ and $(0,1)$, and $\varphi(a,b) = \varphi(1,0)^a \varphi(0,1)^b$, where $\varphi(1,0)$ and $\varphi(0,1)$ are ± 1. Conversely, this formula for φ defines a homomorphism since a and b are in $\mathbf{Z}/(2)$ and exponents on ± 1 only matter mod 2. For φ to be nontrivial means $\varphi(1,0)$ and $\varphi(0,1)$ are not both 1, so there are three choices of $\varphi\colon (\mathbf{Z}/(2))^2 \to (\mathbf{Z}/(3))^\times$:

$$\varphi(a,b) = (-1)^a, \quad \varphi(a,b) = (-1)^b, \quad \varphi(a,b) = (-1)^a(-1)^b = (-1)^{a+b}.$$

This does *not* mean there are corresponding semidirect products $\mathbf{Z}/(3) \rtimes_\varphi (\mathbf{Z}/(2))^2$ are nonisomorphic. In fact, the above three choices of φ lead to isomorphic semidirect products: precomposing the first φ with the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ produces the second φ, and precomposing the first φ with the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ produces the third φ. Therefore the three nontrivial semidirect products $\mathbf{Z}/(3) \rtimes (\mathbf{Z}/(2))^2$ are isomorphic, so all groups of order 12 with $n_2 = 3$ (equivalently, all nonabelian groups of order 12 with $n_3 = 1$) and 2-Sylow subgroup isomorphic to $(\mathbf{Z}/(2))^2$ are isomorphic. One such group is D_6, with normal 3-Sylow subgroup $\{1, r^2, r^4\}$. $\square$

If we meet a group of order 12, we can decide which of the 5 groups it is isomorphic to by the following procedure:

- Is it abelian? If so, it's isomorphic to $\mathbf{Z}/(4) \times \mathbf{Z}(3) \cong \mathbf{Z}/(12)$ or $\mathbf{Z}/(2) \times \mathbf{Z}/(2) \times \mathbf{Z}/(3)$, which are distinguished by the structure of the 2-Sylow subgroup.
- Is it nonabelian with $n_2 = 1$? If so, then it's isomorphic to A_4.
- Is it nonabelian with $n_2 > 1$? If so, then it's isomorphic to D_6 if its 2-Sylow subgroups are noncyclic and it's isomorphic to the nontrivial semidirect product $\mathbf{Z}/(3) \rtimes \mathbf{Z}/(4)$ if its 2-Sylow subgroup is cyclic.

For example, here are four nonabelian groups of order 12:

$$\mathbf{Z}/(2) \times S_3, \quad \mathrm{PSL}_2(\mathbf{F}_3), \quad \mathrm{Aff}(\mathbf{Z}/(6)), \quad \mathrm{Aff}(\mathbf{F}_4).$$

The group $\mathbf{Z}/(2) \times S_3$ has $n_2 > 1$ and its 2-Sylow subgroups are not cyclic, so $\mathbf{Z}/(2) \times S_3 \cong D_6$. The group $\mathrm{PSL}_2(\mathbf{F}_3)$ is nonabelian with $n_2 = 1$, so $\mathrm{PSL}_2(\mathbf{F}_3) \cong A_4$. The group $\mathrm{Aff}(\mathbf{Z}/(6))$ has $n_2 > 1$ and its 2-Sylow subgroups are not cyclic, so $\mathrm{Aff}(\mathbf{Z}/(6)) \cong D_6$. Finally, $\mathrm{Aff}(\mathbf{F}_4)$ is nonabelian with a normal 2-Sylow subgroup, so $\mathrm{Aff}(\mathbf{F}_4) \cong A_4$.

Another way to distinguish the three nonabelian groups of order 12 is to count elements of order 2 in them: D_6 has 7 elements of order 2 (6 reflections and r^3), A_4 has 3 elements of order 2 (the permutations of type $(2,2)$), and $\mathbf{Z}/(3) \rtimes \mathbf{Z}/(4)$ has one element of order 2 (it is $(0,2)$).

In abstract algebra textbooks (not group theory textbooks), $\mathbf{Z}/(3) \rtimes \mathbf{Z}/(4)$ is usually written as T but it is almost never given a name to accompany the label. Should it be called the "obscure group of order 12"? Actually, this group belongs to a standard family of finite groups: the dicyclic groups, also called the binary dihedral groups. They are nonabelian with order $4n$ ($n \geq 2$) and each contains a unique element of order 2. The one of order 8 is Q_8, and more generally the one of order 2^m is the generalized quaternion group Q_{2^m}.