REPRESENTATIONS OF $\text{Aff}(\mathbf{F}_q)$ AND $\text{Heis}(\mathbf{F}_q)$

KEITH CONRAD

For each prime power q, we will construct all irreducible representations over $\mathbf{C}$ of the groups $\text{Aff}(\mathbf{F}_q)$ and $\text{Heis}(\mathbf{F}_q)$. To find all of them, there are three parts:

- Build as many irreducible representations as the number of conjugacy classes.
- Check a representation is irreducible by checking its character has inner product 1 with itself (1-dimensional representations are automatically irreducible).
- Check irreducible representations that are not 1-dimensional are nonisomorphic by checking their characters are different (distinct 1-dimensional representations are automatically nonisomorphic).

1. Representations of $\text{Aff}(\mathbf{F}_q)$

Let F be any field. In $\text{Aff}(F)$, the group law is

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a' & b' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} aa' & ab'+b \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1/a & -b/a \\ 0 & 1 \end{pmatrix}.$$

Two subgroups of $\text{Aff}(F)$ are

$$\left\{\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} : a \neq 0\right\}, \quad \left\{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in F\right\},$$

which are isomorphic to $F^\times$ and F as groups. Matrices in $\text{Aff}(F)$ decompose into a product of elements in these subgroups as

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix},$$

and the formula in alphabetical order doesn't quite work:

$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & ab \\ 0 & 1 \end{pmatrix}.$$

Conjugation in $\text{Aff}(F)$ is

$$\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} a & bx - y(a-1) \\ 0 & 1 \end{pmatrix}$$

In particular, $\text{Aff}(F)$ has trivial center unless $F = \mathbf{F}_2$, in which case $\text{Aff}(F)$ is abelian.

Here are the conjugacy classes in $\text{Aff}(F)$:

- the identity matrix
$$\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\},$$
- the set
$$\left\{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in F^\times\right\},$$

- for each $a \in F$ with $a \neq 0$ and $a \neq 1$, the set
$$\left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : b \in F \right\}.$$

So in $\mathrm{Aff}(\mathbf{F}_q)$ there are a total of $1 + 1 + (q-2) = q$ conjugacy classes and thus there are q irreducible representations of $\mathrm{Aff}(\mathbf{F}_q)$ over $\mathbf{C}$.

One-dimensional representations: Since the upper left entry in $\mathrm{Aff}(\mathbf{F}_q)$ behaves multiplicatively in the group law, for each homomorphism $\chi \colon \mathbf{F}_q^\times \to \mathbf{C}^\times$ we get a one-dimensional representation $\mathrm{Aff}(\mathbf{F}_q) \to \mathbf{C}^\times$ by
$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mapsto \chi(a).$$

Since $\mathbf{F}_q^\times$ is cyclic of order $q-1$, there are $q-1$ such χ, so we get $q-1$ one-dimensional representations of $\mathrm{Aff}(\mathbf{F}_q)$.

Remaining irreducible representation: From the count of conjugacy classes there is one more irreducible representation of $\mathrm{Aff}(\mathbf{F}_q)$. Letting d denote its degree, from $q-1+d^2 = |\mathrm{Aff}(\mathbf{F}_q)| = q(q-1)$ we get $d = q-1$, so we seek a $(q-1)$-dimensional representation.

Consider the complex vector space V of functions $f \colon \mathbf{F}_q \to \mathbf{C}$. This is q-dimensional. Let each $g \in \mathrm{Aff}(\mathbf{F}_q)$ act on V as a linear change of variables using g^{-1}: $(\rho_V(g)f)(x) = f(g^{-1}x)$. We need g^{-1} rather than g in the formula to get $\rho_V(gh) = \rho_V(g)\rho_V(h)$. Explicitly,
$$(\rho_V(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix})f)(x) = f\left(\frac{1}{a}x - \frac{b}{a}\right).$$

The constant functions in V form a one-dimensional subspace on which $\mathrm{Aff}(\mathbf{F}_q)$ acts trivially. Another $\mathrm{Aff}(\mathbf{F}_q)$-stable subspace of V is
$$W = \left\{ f \in V : \sum_{x \in \mathbf{F}_q} f(x) = 0 \right\}$$
and $\rho_V = \rho_W \oplus 1$, where ρ_W is the restriction of ρ_V to W. The dimension of W is $q-1$.

To show W is irreducible, we compute its character from that of V: $\chi_V = \chi_W + 1$. A basis of V is the q delta-functions $\delta_t \colon \mathbf{F}_q \to \mathbf{C}$ for $t \in \mathbf{F}_q$, where $\delta_t(x)$ is 0 for $x \neq t$ and $\delta_t(t) = 1$. Since $\rho_V(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix})\delta_t = \delta_{at+b}$, the matrix for $\rho_V(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix})$ with respect to the delta-basis of V is a permutation matrix based describing how $t \mapsto at + b$ permutes $\mathbf{F}_q$. Thus
$$\chi_V \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \#\{t \in \mathbf{F}_q : at + b = t \text{ in } \mathbf{F}_q\} = \begin{cases} 1, & \text{if } a \neq 1, \\ q, & \text{if } a = 1 \text{ and } b = 0, \\ 0, & \text{if } a = 1 \text{ and } b \neq 0. \end{cases}$$

Therefore
$$\chi_W \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \chi_V \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} - 1 = \begin{cases} 0, & \text{if } a \neq 1, \\ q-1, & \text{if } a = 1 \text{ and } b = 0, \\ -1, & \text{if } a = 1 \text{ and } b \neq 0. \end{cases}$$

The inner product of χ_W with itself is
$$\frac{1}{q(q-1)} \sum_g \chi_W(g)\overline{\chi_W(g)} = \frac{1}{q(q-1)}((q-1)^2 + (q-1)(-1)^2) = \frac{(q-1)^2 + (q-1)}{q(q-1)} = 1,$$

so W is irreducible. It is a new irreducible representation since it's not 1-dimensional, except if $q = 2$, in which case ρ_W is nontrivial while the single one-dimensional representation constructed earlier ($q-1 = 1$ if $q = 2$) is trivial (note $\mathrm{Aff}(\mathbf{F}_2) \cong \mathbf{F}_2$).

2. Representations of $\mathrm{Heis}(\mathbf{F}_q)$

For a field F, the group law in the Heisenberg group $\mathrm{Heis}(F)$ is

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & a' & b' \\ 0 & 1 & c' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+a' & b+b'+ac' \\ 0 & 1 & c+c' \\ 0 & 0 & 1 \end{pmatrix} \tag{1}$$

with inverse formula

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -a & -b+ac \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{pmatrix}.$$

Three subgroups of $\mathrm{Heis}(F)$ are

$$\left\{\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : a \in F\right\}, \quad \left\{\begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : b \in F\right\}, \quad \left\{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : c \in F\right\},$$

which are each isomorphic to F as groups. Note the subset

$$\left\{\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, c \in F\right\}$$

is *not* a subgroup of $\mathrm{Heis}(F)$ since it's not closed under multiplication.

Matrices in $\mathrm{Heis}(F)$ decompose into a product of matrices in the three subgroups as

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2}$$

while it doesn't quite work in alphabetical order:

$$\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a & b+ac \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}.$$

Conjugation in $\mathrm{Heis}(F)$ is

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & a & b-az+cx \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}.$$

In particular, the center of $\mathrm{Heis}(F)$ is

$$\left\{\begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : b \in F\right\}.$$

Here are the conjugacy classes in $\mathrm{Heis}(F)$:

- for each $b \in F$, the single matrix
$$\left\{\begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\},$$
- for each pair $(a,c) \in F^2 - \{(0,0)\}$, the set
$$\left\{\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : b \in F\right\}.$$

When $F = \mathbf{F}_q$, there are $q + (q^2 - 1) = q^2 + q - 1$ conjugacy classes, and thus $q^2 + q - 1$ irreducible representations of $\mathrm{Heis}(\mathbf{F}_q)$ over $\mathbf{C}$.

One-dimensional representations: Let $\psi\colon \mathbf{F}_q \to \mathbf{C}^\times$ be a nontrivial homomorphism. An example is $\psi(x) = e^{2\pi i \mathrm{Tr}_{\mathbf{F}_q/\mathbf{F}_p}(x)}$, where $\mathbf{F}_q$ has characteristic p and $\mathrm{Tr}_{\mathbf{F}_q/\mathbf{F}_p}\colon \mathbf{F}_q \to \mathbf{F}_p$ is the trace map. (If $q = p$ then $\psi\colon \mathbf{F}_p \to \mathbf{C}^\times$ by $\psi(x) = e^{2\pi i x/p}$.) Since the a and c terms of a matrix in $\mathrm{Heis}(\mathbf{F}_q)$ combine additively under matrix multiplication, for each $(a,c) \in \mathbf{F}_q^2$ there is a representation $\psi_{a,c}\colon \mathrm{Heis}(\mathbf{F}_q) \to \mathbf{C}^\times$ given by
$$\psi_{a,c}\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} = \psi(ax + cz).$$
Check as an exercise that $\psi_{a,c}$ changes if the pair (a,c) changes, so we have q^2 irreducible representations of degree 1.

Remaining irreducible representations: We need $q-1$ more irreducible representations, and their degrees $\{d_i\}$ must satisfy $q^2 + \sum d_i^2 = |\,\mathrm{Heis}(\mathbf{F}_q)| = q^3$, so $\sum d_i^2 = q^3 - q^2 = (q-1)q^2$. We will find $q-1$ nonisomorphic irreducible representations with degree q.

Let V, as before, be the q-dimensional vector space of functions $f\colon \mathbf{F}_q \to \mathbf{C}$. We will define three actions of $\mathbf{F}_q$ on V, and composing them in a suitable order will give an action of the group $\mathrm{Heis}(\mathbf{F}_q)$ on V. For $a,b,c \in \mathbf{F}_q$, define the linear maps $\sigma_a\colon V \to V$, $\tau_b\colon V \to V$, and $\varphi_c\colon V \to V$ by
$$(\sigma_a f)(x) = f(x+a), \quad (\tau_b f)(x) = \psi(b)f(x), \quad (\varphi_c f)(x) = \psi(cx)f(x),$$
where $\psi\colon \mathbf{F}_q \to \mathbf{C}^\times$ is the same function we used to build one-dimensional representations above. For $t \in \mathbf{F}_q$, note τ_t multiplies each function in V by the *number* $\psi(t)$ while φ_t multiplies each function in V by the *function* $\psi(tx)$.

Check $\sigma_a \circ \sigma_{a'} = \sigma_{a+a'}$, $\tau_b \circ \tau_{b'} = \tau_{b+b'}$, and $\varphi_c \circ \varphi_{c'} = \varphi_{c+c'}$ as functions $V \to V$, so $a \mapsto \sigma_a$, $b \mapsto \tau_b$, $c \mapsto \varphi_c$ are actions of $\mathbf{F}_q$ on V by linear maps. Check τ_b commutes with both σ_a and φ_c, while σ_a and φ_c commute with each other up to scaling by a root of unity:
$$\sigma_a \circ \tau_b = \tau_b \circ \sigma_a, \quad \varphi_c \circ \tau_b = \tau_b \circ \varphi_c, \quad \sigma_a \circ \varphi_c = \psi(ac)\varphi_c \circ \sigma_a = \tau_{ac} \circ \varphi_c \circ \sigma_a. \tag{3}$$
Using (3), the 3-fold composites $\sigma_a \circ \tau_b \circ \varphi_c$ compose as follows:
$$\begin{aligned}(\sigma_a \circ \tau_b \circ \varphi_c) \circ (\sigma_{a'} \circ \tau_{b'} \circ \varphi_{c'}) &= \sigma_a \circ \tau_b \circ (\varphi_c \circ \sigma_{a'}) \circ \tau_{b'} \circ \varphi_{c'} \\ &= \sigma_a \circ \tau_b \circ (\tau_{-a'c} \circ \sigma_{a'} \circ \varphi_c) \circ \tau_{b'} \circ \varphi_{c'} \\ &= \sigma_a \circ \tau_{b-a'c} \circ \sigma_{a'} \circ \tau_{b'} \circ \varphi_c \circ \varphi_{c'} \\ &= \sigma_a \circ \sigma_{a'} \circ \tau_{b-a'c} \circ \tau_{b'} \circ \varphi_c \circ \varphi_{c'} \\ &= \sigma_{a+a'} \circ \tau_{b+b'-a'c} \circ \varphi_{c+c'}.\end{aligned}$$

This is almost like the way matrices in $\mathrm{Heis}(\mathbf{F}_q)$ multiply in (1), and it would match matrix multiplication if $\tau_{b+b'-a'c}$ were $\tau_{b+b'+ac'}$. Matrices in $\mathrm{Heis}(\mathbf{F}_q)$ decompose in (2) using reverse alphabetical order, which suggests looking at $\varphi_c \circ \tau_b \circ \sigma_a$ instead. By calculations as above,

$$\begin{aligned}
(\varphi_c \circ \tau_b \circ \sigma_a) \circ (\varphi_{c'} \circ \tau_{b'} \circ \sigma_{a'}) &= \varphi_c \circ \tau_b \circ (\sigma_a \circ \varphi_{c'}) \circ \tau_{b'} \circ \sigma_{a'} \\
&= \varphi_c \circ \tau_b \circ (\tau_{ac'} \circ \varphi_{c'} \circ \sigma_a) \circ \tau_{b'} \circ \sigma_{a'} \\
&= \varphi_c \circ \tau_{b+ac'} \circ \varphi_{c'} \circ \tau_{b'} \circ \sigma_a \circ \sigma_{a'} \\
&= \varphi_c \circ \varphi_{c'} \circ \tau_{b+ac'} \circ \tau_{b'} \circ \sigma_a \circ \sigma_{a'} \\
&= \varphi_{c+c'} \circ \tau_{b+b'+ac'} \circ \sigma_{a+a'}.
\end{aligned}$$

This is exactly how matrices in $\mathrm{Heis}(\mathbf{F}_q)$ multiply, so we get a q-dimensional representation ρ_ψ of $\mathrm{Heis}(\mathbf{F}_q)$ on V by

$$g = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mapsto \rho_\psi(g) = \varphi_c \circ \tau_b \circ \sigma_a.$$

As a formula, for $f \in V$

$$(\rho_\psi(g)f)(x) = (\varphi_c\tau_b\sigma_a f)(x) = \psi(cx)(\tau_b\sigma_a f)(x) = \psi(cx)\psi(b)f(x+a) = \psi(cx+b)f(x+a).$$

Let's compute the character of ρ_ψ. Using the basis $\{\delta_t : t \in \mathbf{F}_q\}$ of V, we have

$$g = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \Longrightarrow (\rho_\psi(g)\delta_t)(x) = \psi(cx+b)\delta_t(x+a) = \begin{cases} \psi(c(t-a)+b), & \text{if } x = t-a, \\ 0, & \text{if } x \neq t-a, \end{cases}$$

so

$$\rho_\psi(g)\delta_t = \psi(c(t-a)+b)\delta_{t-a}.$$

- If $a \neq 0$, the matrix of $\rho_\psi(g)$ with respect to the delta-basis has 0's on the main diagonal, so $(\mathrm{Tr}\ \rho_\psi)(g) = 0$.
- If $a = 0$, the matrix of $\rho_\psi(g)$ with respect to the delta-basis has diagonal entries $\psi(ct+b)$, so $(\mathrm{Tr}\ \rho_\psi)(g) = \sum_{t \in \mathbf{F}_q} \psi(ct+b)$, which is $q\psi(b)$ if $c = 0$ and 0 if $c \neq 0$.

Thus the character χ_ψ of ρ_ψ is

$$\chi_\psi \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} = \begin{cases} q\psi(b), & \text{if } a = c = 0, \\ 0, & \text{otherwise} \end{cases}$$

so the inner product of χ_ψ with itself is

$$\frac{1}{q^3}\sum_g \chi_\psi(g)\overline{\chi_\psi(g)} = \frac{1}{q^3}\sum_{b \in \mathbf{F}_q} q\psi(b)\overline{q\psi(b)} = \frac{1}{q}\sum_{b \in \mathbf{F}_q} 1 = 1.$$

We have a q-dimensional irreducible representation ρ_ψ for each nontrivial homomorphism $\psi\colon \mathbf{F}_q \to \mathbf{C}^\times$. There are $q-1$ choices for ψ. To prove the ρ_ψ's for different ψ are nonisomorphic, we show the characters of the ρ_ψ's are different. This follows from the formula

$$\psi(b) = \frac{1}{q}\chi_\psi \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which shows we can recover ψ from the character of ρ_ψ. (Alternatively, check for different nontrivial ψ_1 and ψ_2 that the inner product of the characters of ρ_{ψ_1} and ρ_{ψ_2} is 0.)

More on the irreducible representations of $\mathrm{Aff}(\mathbf{F}_q)$ and $\mathrm{Heis}(\mathbf{F}_q)$ is in [1, Chap. 16–18].

References

[1] A. Terras, "Fourier Analysis on Finite Groups and Applications," Cambridge Univ. Press, 1999.