NO SUBGROUP OF A_4 HAS INDEX 2

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The group A_4 has order 12, so its subgroups could have size 1, 2, 3, 4, 6, or 12. There are subgroups of orders 1, 2, 3, 4, and 12, but A_4 has no subgroup of order 6 (equivalently, no subgroup of index 2). Here is one proof, using left cosets.

Theorem 1. *There is no subgroup of index* 2 *in* A_4*.*

Proof. Suppose a subgroup H of A_4 has index 2, so $|H| = 6$. We will show for each $g \in A_4$ that $g^2 \in H$.

If $g \in H$ then clearly $g^2 \in H$. If $g \notin H$ then gH is a left coset of H different from H (since $g \in gH$ and $g \notin H$), so from $[G : H] = 2$ the only left cosets of H are H and gH. Which one is g^2H? If $g^2H = gH$ then $g^2 \in gH$, so $g^2 = gh$ for some $h \in H$, and that implies $g = h$, so $g \in H$, but that's a contradiction. Therefore $g^2H = H$, so $g^2 \in H$.

Every 3-cycle (abc) in A_4 is a square: (abc) has order 3, so $(abc) = (abc)^4 = ((abc)^2)^2$. Thus H contains all 3-cycles in A_4. The 3-cycles are

$$(123),\ (132),\ (124),\ (142),\ (134),\ (143),\ (234),\ (243)$$

and that is too much since there are 8 of them while $|H| = 6$. Hence H does not exist. □

We will now give three more proofs that there is no subgroup of index 2 in A_4 as corollaries of three different theorems from group theory.

Theorem 2. *If G is a finite group and $N \lhd G$ then any element of G with order relatively prime to $[G : N]$ lies in N. In particular, if N has index* 2 *then all elements of G with odd order lie in N.*

Proof. Let g be an element of G with order m, which is relatively prime to $[G : N]$. Reducing the equation $g^m = e$ modulo N gives $\overline{g}^m = \overline{e}$ in G/N. Also $\overline{g}^{[G:N]} = \overline{e}$, so the order of $\overline{g}$ in G/N divides m and $[G : N]$. These numbers are relatively prime, so $\overline{g} = \overline{e}$, which means $g \in N$. □

Corollary 3. *There is no subgroup of index* 2 *in* A_4*.*

Proof. If A_4 has a subgroup with index 2 then by Theorem 2, all elements of A_4 with odd order are in the subgroup. But A_4 contains 8 elements of order 3 (there are 8 different 3-cycles), and an index-2 subgroup of A_4 has size 6, so not all elements of odd order can lie in the subgroup. □

That proof is very closely related to the first proof we gave.

Theorem 4. *If G is a finite group with a subgroup of index* 2 *then its commutator subgroup has even index.*

Proof. If $[G : H] = 2$ then $H \lhd G$, so G/H is a group of size 2 and thus is abelian. So all commutators of G are in H, which means H contains the commutator subgroup of G. The index of the commutator subgroup therefore is divisible by $[G : H] = 2$. □

Corollary 5. *There is no subgroup of index* 2 *in* A_4*.*

Proof. We will show the commutator subgroup of A_4 has odd index, so there is no index-2 subgroup by Theorem 4. The subgroup

$$V = \{(1), (12)(34), (13)(24), (14)(23)\}$$

is normal in A_4 and A_4/V has size 3, hence is abelian, so the commutator subgroup of A_4 is inside V. Each element of V is a commutator (*e.g.*, $(12)(34) = [(123), (124)]$), so V is the commutator subgroup of A_4. It has index 3, which is odd. □

Theorem 6. *Every group of size* 6 *is cyclic or isomorphic to* S_3*.*

Proof. This is a special case of the classification of groups of order pq for primes p and q, but we give a self-contained treatment in this special case.

Let G have size 6 and assume G is not cyclic. We want to show $G \cong S_3$. By Cauchy, G contains elements a with order 2 and b with order 3. The subgroup $H = \{1, a\}$ has index 3, so the usual left multiplication action of G on the left coset space G/H is a homomorphism $G \to \mathrm{Sym}(G/H) \cong S_3$. If g is in the kernel then $gH = H$, so $g \in H$. Thus, if the kernel is nontrivial then it contains a. In particular, $abH = bH$. Since $bH = \{b, ba\}$ and $abH = \{ab, aba\}$, either $b = ab$ or $b = aba$. The first choice is impossible, so $b = aba$. Since a has order 2, $ab = ba^{-1} = ba$, which means a and b commute. Thus ab has order $2 \cdot 3 = 6$, so G is cyclic. We were assuming G is not cyclic, so the kernel of the map $G \to \mathrm{Sym}(G/H)$ is trivial, hence this is an isomorphism. □

Corollary 7. *There is no subgroup of index* 2 *in* A_4*.*

Proof. If A_4 has an index-2 subgroup H, that subgroup has size 6 and therefore is isomorphic to either $\mathbf{Z}/(6)$ or S_3. There are no elements in A_4 with order 6, so the first choice is impossible: H must be isomorphic to S_3. In S_3 there are three elements of order 2 (the transpositions). The group A_4 also has only three elements of order 2 ($(12)(34), (13)(24), (14)(23)$), so these $(2, 2)$-cycles must lie in H. However, the elements of order 2 in S_3 don't commute while the $(2, 2)$-cycles in A_4 do commute, so we have a contradiction. Since H can't be isomorphic to S_3, it doesn't exist. □

For more proofs of this result, see [1].

References

[1] M. Brennan, D. Machale, Variations on a theme: A_4 definitely has no subgroup of order six!, *Math. Mag.* **73** (2000), 36–40.