THE 15-PUZZLE (AND RUBIK'S CUBE)

KEITH CONRAD

1. Introduction

A permutation puzzle is a toy where the pieces can be moved around and the object is to reassemble the pieces into their beginning state We will discuss two such puzzles: the 15-puzzle and Rubik's Cube. Our analysis of the 15-puzzle will be complete, but we will only be able to sketch some basic ideas behind the mathematics of Rubik's Cube.

2. The 15-puzzle

The 15-puzzle contains 15 sliding pieces and one empty square. It looks like this:

(2.1)

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

After sliding the pieces around until they are jumbled pretty thoroughly, the object of the puzzle is to bring it back to the arrangement above.

Actually, this is not the way the puzzle was first introduced. It was originally constructed with 15 *removable* blocks and the challenge was to start with the blocks in the arrangement

(2.2)

1	2	3	4
5	6	7	8
9	10	11	12
13	15	14	

where 14 and 15 are switched. (Since the original puzzle had removable blocks, this configuration of blocks could be obtained by taking pieces out.) Now slide pieces around – with no removals – in order to bring the arrangement from (2.2) to the natural order (2.1).

Before discussing the mathematics of the puzzle, here is a brief history of it. It was created in the 1870s in New England, and was slow to catch on. Then in 1880, the puzzle swept very quickly across America and Europe. Many people came forward announcing they could solve it, but either they were unable to demonstrate their winning sequence of moves in public or they misunderstood the challenge itself. Starting in the 1890s, Sam Loyd offered a \$1000 prize for anyone who could show a solution, and it is commonly believed that Loyd invented the puzzle, but that is false.[1]

The 15-puzzle is sold today in a plastic casing, so the sliding pieces can't easily be removed. We can still consider the original challenge of the puzzle, just in (equivalent) reverse order: can one start with (2.1) and obtain (2.2)?

Theorem 2.1. *It is impossible to pass between* (2.1) *and* (2.2) *by sliding the pieces.*

[1] See [4] for more history on this puzzle.

Proof. Going from (2.1) to (2.2) or *vice versa* are equivalent problems. We'll focus on the impossibility of going from (2.1) to (2.2).

The key observation is that each move in the puzzle *must* involve switching the empty space and some other (adjacent) piece. Since we have to keep track of the empty space, let's give it a name, say 16:

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 & 8 \\ \hline 9 & 10 & 11 & 12 \\ \hline 13 & 14 & 15 & \mathbf{16} \\ \hline \end{array} \tag{2.3}$$

We write 16 in bold in the diagram to remind us of its distinguished role.

A sequence of moves is a composition of several transpositions involving 16. For instance, starting from the configuration as in (2.3), we can do a clockwise rotation of the pieces 11, 12, and 15 as follows:

$$\begin{array}{|c|c|}\hline 11 & 12 \\ \hline 15 & \\ \hline \end{array} \rightsquigarrow \begin{array}{|c|c|}\hline 11 & \\ \hline 15 & 12 \\ \hline \end{array} \rightsquigarrow \begin{array}{|c|c|}\hline & 11 \\ \hline 15 & 12 \\ \hline \end{array} \rightsquigarrow \begin{array}{|c|c|}\hline 15 & 11 \\ \hline & 12 \\ \hline \end{array} \rightsquigarrow \begin{array}{|c|c|}\hline 15 & 11 \\ \hline 12 & \\ \hline \end{array} \tag{2.4}$$

Labeling the blank space as 16, this sequence of moves is the product of transpositions

$$(12\ 16)(15\ 16)(11\ 16)(12\ 16). \tag{2.5}$$

While we read (2.4) from left to right, keep in mind that composition of permutations in (2.5) is read from right to left, since we are following the usual convention of composing functions from right to left.

Being able to create within the 15-puzzle a permutation that turns (2.1) into (2.2) amounts to expressing the transposition (14 15) as a product of transpositions involving 16:

$$(14\ 15) = (a_n\ 16)(a_{n-1}\ 16)\cdots(a_2\ 16)(a_1\ 16), \tag{2.6}$$

where $1 \leq a_i \leq 15$. Because 16 returns to its original location in the bottom right square, it had to move up and down an equal number of times, and right and left an equal number of times. That is, the number of transpositions on the right side of (2.6) is *even*. Therefore the right side is a product of an even number of transpositions, but the left side has an odd number of transpositions. This is a contradiction, so we are done. □

Corollary 2.2. *Any physical rearrangement of pieces in the* 15*-puzzle starting from the standard configuration* (2.3) *that brings the empty space back to its original position must be an even permutation of the other* 15 *pieces.*

Proof. When the empty space returns to its original position, we can view the overall result as a permutation $\pi \in S_{15}$ on the other 15 pieces. Run through the proof of Theorem 2.1, with (14 15) replaced by π. This basically means use π in place of (14 15) on the left side of (2.6). The right side of (2.6) is an even permutation in S_{16} since the blank space 16 is moved an even number of positions, so π is even as a permutation in S_{16}. The parity of a permutation in S_{15} is the same as its parity when viewed as a permutation in S_{16}, so π is an even permutation of the pieces $1, 2, \ldots, 15$. □

The number of permutations of 15 objects is $15! = 1307674368000$. The number of even permutations of 15 objects is

$$\frac{15!}{2} = 653837184000.$$

The parity constraint in Corollary 2.2 tells us $15!/2$ is an upper bound on the number of (legal) positions of the pieces in the 15-puzzle when the empty space returns to the lower right corner. Is this bound achieved? Equivalently, can every permutation in A_{15} be expressed as a product of transpositions involving 16 that returns 16 back to its initial location? Yes. We will use the 3-cycles in A_{15} to complete this analysis.

Theorem 2.3. *For $n \geq 3$, A_n is generated by the 3-cycles $(12i)$.*

This is a standard result in group theory and we omit the proof. It is a refinement of the more widely familiar fact in group theory that A_n is generated by *all* 3-cycles when $n \geq 3$.

By Corollary 2.2, the group of configurations of the 15-puzzle that can be reached from the configuration (2.1) and have the empty space in the lower-right position, is a subgroup of A_{15}. We want to show the subgroup is all of A_{15}. Our key tool will be Theorem 2.3 in a "coordinate-free" form: A_{15} is generated by the 3-cycles involving a fixed common pair of terms. We will take these to be the 3-cycles $(11\ 12\ i)$ instead of $(12i)$.

We saw in (2.4) that the 3-cycle $(11\ \ 12\ \ 15)$ is possible. Let m denote the total move in (2.4). If, from a solved puzzle, we can find a sequence of moves $g_i \in S_{16}$ to carry any piece $i \neq \{11, 12, 15, 16\}$ into the space naturally occupied by 15 without moving 11, 12, or 16, then the 3-cycle $(11\ 12\ i)$ is $g_i^{-1} m g_i$ (read from right to left) and we'd be done.

Start by moving the empty space to the inside of the puzzle by exchanging it with 12 and then 11:

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

$\rightsquigarrow$

1	2	3	4
5	6	7	8
9	10		11
13	14	15	12

Now we produce two 'tours' of the rest of the board that pass through the empty space and the 15 in the configuration on the right side above. See the figures below, where each path is highlighted in bold and we use 16 as a label for the empty space. These paths are: 16,7,3,2,1,5,9,13,14,15 and 16,7,8,4,3,2,6,10,14,15.

1	**2**	**3**	4
5	6	**7**	8
9	10	**16**	11
13	**14**	**15**	12

1	**2**	**3**	**4**
5	**6**	**7**	**8**
9	**10**	**16**	11
13	**14**	**15**	12

(I found these paths in [3, pp. 123–124], which is all about permutation puzzles.) In case you don't believe these paths will take each highlighted piece into 15's position, simply find a 15-puzzle and try it.

This completes the analysis of the 15-puzzle: exactly half the arrangements of the tiles 1 through 15 can be reached from the standard position, since A_{15} is generated by 3-cycles with a common pair of terms.

Let's look at an example of a random rearrangement of the pieces and see if it can be reached from (2.1):

8	7	6	5
9	3	1	10
2	11	14	4
12	15	13	

Comparing this to (2.1), this rearrangement is the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 8 & 7 & 6 & 5 & 9 & 3 & 1 & 10 & 2 & 11 & 14 & 4 & 12 & 15 & 13 \end{pmatrix},$$

which when written as a product of disjoint cycles becomes

$$(1\ 8\ 10\ 11\ 14\ 15\ 13\ 12\ 4\ 5\ 9\ 2\ 7)(3\ 6).$$

This is a 13-cycle times a 2-cycle, which is an even permutation (13-cycle) times an odd permutation (2-cycle), so overall this rearrangement is an odd permutation. Therefore it is impossible to reach this state from (2.1), or conversely to go from this state to (2.1).

3. Rubik's Cube

Nothing like the 19-th century frenzy over the 15-puzzle was seen again until essentially 100 years later, when Rubik's Cube came on the scene in the early 1980s. Its inventor, Erno Rubik, became the first self-made millionaire in the Communist bloc.

It's best if you have a copy of the cube to play with as you read the remaining discussion. We will *not* describe a solution to the cube, although you can find some on the internet that don't require too much memorization. (Such a method in book form is in [2, Chapter 3].) What we will do here is introduce enough notation and terminology to explain what the the group of all permutations of Rubik's Cube is, much like the group of all permutations of the 15 puzzle (preserving the empty square in the bottom right corner) is A_{15}.

If you use a screwdriver to carefully pop out a piece along an edge (see Figures 1 and 2) then the rest of the pieces easily come out and the interesting center mechanism is revealed (Figure 3). This shows a basic fact about the cube: the 6 center pieces are actually one single piece and no amount of turning will ever change the relative positions of the center faces. Because the center pieces always maintain the same relative positions, each central color tells you what color that whole face must be in the solved cube. For instance, if a messed up cube has blue and green as opposite center colors then the solved state of that cube will have blue and green faces opposite each other.

Figure 1. Beginning to disassemble the cube along an edge.

There are three kinds of pieces in the cube: 8 corner pieces (each with 3 stickers), 12 edge pieces (each having 2 stickers) and 6 center pieces (each with one sticker). See Figure 4. The number of non-center stickers is $8 \cdot 3 + 12 \cdot 2 = 48$. When you make a move of the cube, the 3 colors on a corner stay together and the 2 colors on an edge stay together.

Although in practice you might want to physically rotate the entire cube in space to get a better view of a face, this rotation is not a move at all: relative positions of every piece stay the same under a rotation. So in order to discuss constraints on what can be done with Rubik's Cube, we can always keep the center pieces in fixed positions (no cube rotations).

FIGURE 2. One edge piece out.

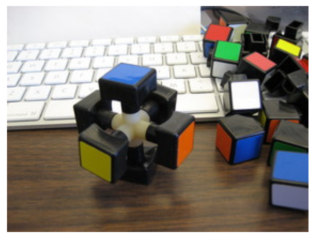

FIGURE 3. The center mechanism.

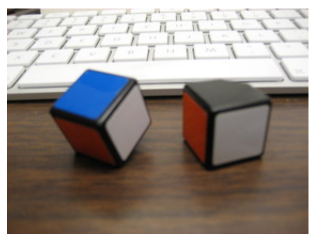

FIGURE 4. A corner and edge piece.

When holding the cube with one face facing you, the labels of the 6 faces are

- F for Front,
- B for Back,
- L for Left,
- R for Right,
- U for Up,
- D for Down.

See Figure 5. The labels Up/Down are used instead of Top/Bottom to avoid the confusion over the meaning of B (Bottom or Back?) I have seen a book on solving Rubik's Cube that does use the labels Top/Bottom, and calls the Back face the P(osterior) face, but this is pretty uncommon. The face labels we use here, due to D. Singmaster, are essentially universally accepted[2].

[2]The colors on the cube are not universally standardized among different manufacturers. Even cubes with the same 6 face colors can have them appear in different positions: white may be opposite blue on one solved cube but be opposite red on another solved cube. This is why it is important to refer to arrangements of pieces on the cube using a notation that is color independent, like Singmaster's notation.

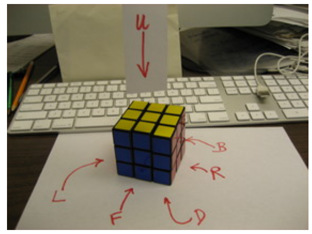

FIGURE 5. Face Names.

The face labels F, B, L, R, U, D are used in two ways: to mark the center cube on each face (which does not move), and also to denote a quarter-turn clockwise rotation on that face if you look at the face head-on in a natural way. What this means in terms of a cube you are holding with F in front of you (and U lying above it) is:

- F is a quarter-turn of the Front face carrying its top row to R,
- B is a quarter-turn of the Back face carrying its top row to L,
- L is a quarter-turn of the Left face carrying its top row to F,
- R is a quarter-turn of the Right face carrying its top row to B,
- U is a quarter-turn of the Up face carrying its front row to L,
- D is a quarter-turn of the Down face carrying its front row to R.

We call these 6 quarter-turns the *basic* moves of the cube. There is another natural class of moves: quarter-turns of the three middle layers in the cube. These can be accounted for with our basic moves since a quarter-turn in one direction of any middle layer has the same effect on the cube as quarter-turns in the opposite direction of the two parallel outer layers, which amounts to a product of two of the six basic moves above.

Below is a diagram of the cube if we imagine unfolding the faces. (The numbers $1, 2, \ldots, 48$ correspond to each non-center sticker.)

			1	2	3						
			4	U	5						
			6	7	8						
9	10	11	17	18	19	25	26	27	33	34	35
12	L	13	20	F	21	28	R	29	36	B	37
14	15	16	22	23	24	30	31	32	38	39	40
			41	42	43						
			44	D	45						
			46	47	48						

Referring to the cube-face diagram above, a tedious verification shows the 6 basic moves are the following elements of S_{48}:

$$\begin{aligned}
\mathrm{F} &= (17,19,24,22)(18,21,23,20)(6,25,43,16)(7,28,42,13)(8,30,41,11),\\
\mathrm{B} &= (33,35,40,38)(34,37,39,36)(3,9,46,32)(2,12,47,29)(1,14,48,27),\\
\mathrm{L} &= (9,11,16,14)(10,13,15,12)(1,17,41,40)(4,20,44,37)(6,22,46,35),\\
\mathrm{R} &= (25,27,32,30)(26,29,31,28)(3,38,43,19)(5,36,45,21)(8,33,48,24),\\
\mathrm{U} &= (1,3,8,6)(2,5,7,4)(9,33,25,17)(10,34,26,18)(11,35,27,19),\\
\mathrm{D} &= (41,43,48,46)(42,45,47,44)(14,22,30,38)(15,23,31,39)(16,24,32,40).
\end{aligned}$$

From a group-theoretic perspective, understanding all possible configurations of a Rubik's Cube amounts to asking: what subgroup of S_{48} is generated by F, B, L, R, U, D:

$$\langle \mathrm{F, B, L, R, U, D} \rangle = ???.$$

This set of all products of permutations generated by the 6 moves is called Rubik's group. Can it be written down in terms of simpler known groups? This is comparable to the connection between the arrangements of the pieces in the 15-puzzle and the group A_{15}.

Since corner and edge pieces can never occupy each other's positions, thinking about Rubik's group inside S_{48} is not such a great idea. We should consider the corner and edge pieces separately. However, although each move of the cube permutes the 8 corner pieces among themselves and the 12 edge pieces among themselves, there is more information in a move than how it permutes the corner pieces and how it permutes the edge pieces: each corner and edge piece has an *orientation*, describing how it fits into its current position.

We call any position that a corner or edge piece can be placed in a *cubicle*. There are 20 of them: 8 corner cubicles and 12 edge cubicles. A corner cubicle can be filled by a corner piece in 3 ways, while an edge cubicle can be filled by an edge piece in 2 ways. These different possibilities are called the orientations of the (corner or edge) piece. We call the pieces in the solved state of the cube 'oriented.' How can we decide if the pieces are oriented or not in any other state of the cube?

Each corner piece has a color matching the center color of the U or D face. Mark that face of the corner. On the edge pieces having a color belonging to U or D, mark that face of the edge. On the edge pieces not having a color belonging to U or D, there will be a color belonging to F or B. Mark the face with that color. We have marked one face of each corner piece and each edge piece.

Now play with the cube, remembering not to change the location of the center pieces (that is, don't rotate the whole cube in space). When the pieces are scattered about the cube, assign a corner piece and edge piece an *orientation* value that is in $\mathbf{Z}/(3)$ for corners and in $\mathbf{Z}/(2)$ for edges according to the following rules:

- If a corner piece has its marked color on the U or D face, we declare the piece to have orientation value 0. (A corner piece will never be in the middle layer.) For any corner piece that has its marked color on a different face, that color can be brought to the U or D face by a $1/3$ rotation (in your mind) either clockwise or counterclockwise. If we can bring the marked color to the U or D face with a clockwise $1/3$ rotation, give the piece orientation value 1. Otherwise we can bring the marked color to the U or D face with a counterclockwise $1/3$ rotation and we give the piece orientation value -1. (Thus, in all cases, an orientation value of n

on a corner piece means a clockwise rotation by $2n\pi/3$ radians will put the marked color of the piece on the U or D face.)
- If an edge piece is in the upper or lower layer of the cube and has its marked color on the U or D face, give the piece orientation value 0. If the piece is in the middle layer and its marked color is on the F or B face, give the piece orientation value 0. In other cases we assign the piece orientation value 1.

Instead of viewing a move of the cube in S_{48} (as a permutation of the stickers) we can view it as a permutation of the 8 corner pieces, keeping track of the 3 orientation values for each corner piece, and a permutation of the 12 edge pieces, keeping track of the 2 orientations of each corner piece. (That is still $8 \times 3 + 12 \times 2 = 48$ pieces of information.) Give the corner pieces a definite labeling $1, 2, \dots, 8$ and the edge pieces a definite labeling $1, 2, \dots, 12$. Then any move of the cube corresponds to a choice of 4-tuple from

$$S_8 \times S_{12} \times (\mathbf{Z}/(3))^8 \times (\mathbf{Z}/(2))^{12}. \tag{3.1}$$

Which 4-tuples $(\pi, \rho, \mathbf{v}, \mathbf{w})$ from this set really correspond to moves on the cube? There are a few constraints. First of all, as a permutation on the pieces, each move among F, B, L, R, U, D is a 4-cycle on 4 corner pieces and a 4-cycle on 4 edge pieces. A 4-cycle is odd, so each basic move gives an odd permutation in S_8 and in S_{12}. This might sound strange: odd permutations do not form a group! However, let's think about the fact that *both* of the permutations of corner and edge pieces in any of F, B, L, R, U, D are odd. When composed, permutations with this feature will have both odd or both even effects on the corner and edge pieces. In other words, any two permutations $\pi \in S_8$ and $\rho \in S_{12}$ coming from a move of the cube satisfy

$$\operatorname{sgn}(\pi) = \operatorname{sgn}(\rho). \tag{3.2}$$

As for the orientations, a computation shows that each basic move does not change the sum of the coordinates in the orientation vectors $\mathbf{v}$ and $\mathbf{w}$ for a particular arrangement of the pieces. Thus, since a solved cube has both orientation vectors equal to $\mathbf{0}$, any actual move of the cube must have

$$\sum_{i=1}^{8} v_i \equiv 0 \bmod 3, \quad \sum_{j=1}^{12} w_j \equiv 0 \bmod 2. \tag{3.3}$$

(The first formula in (3.3) tells us that in any move the cube, we can't change the orientation of a single corner piece without changing something else. Similarly, the second formula in (3.3) tells us no move of the cube can change the orientation of a single edge piece without changing something else. A single corner rotation would change $\sum_{i=1}^{8} v_i \bmod 3$ by 1, which doesn't preserve the condition $\sum_{i=1}^{8} v_i \equiv 0 \bmod 3$.)

The conditions (3.2) and (3.3) carve out the following subset of (3.1):

$$\left\{ (\pi, \rho, \mathbf{v}, \mathbf{w}) : \operatorname{sgn}\pi = \operatorname{sgn}\rho,\ \sum_{i=1}^{8} v_i \equiv 0 \bmod 3,\ \sum_{j=1}^{12} w_j \equiv 0 \bmod 2 \right\}. \tag{3.4}$$

Every arrangement of the pieces in Rubik's Cube that can be reached from the solved state lies in (3.4). It turns out that, conversely, every 4-tuple in (3.4) is a solvable arrangement of the pieces in Rubik's Cube. This is shown in [1, p. 42], which gives an (inefficient) algorithm to solve the cube starting from any position satisfying (3.4). Therefore the number of arrangements of the pieces in Rubik's Cube is the size of (3.4). How large is (3.4)? Among all pairs of permutations $(\pi, \rho) \in S_8 \times S_{12}$, half have $\operatorname{sgn} \pi = \operatorname{sgn} \rho$. Among the 8-tuples $\mathbf{v} \in (\mathbf{Z}/(3))^8$, one-third have the sum of coordinates equal to 0. Among the 12-tuples $\mathbf{w} \in (\mathbf{Z}/(2))^{12}$, half have the sum of coordinates equal to 0. So the total number of arrangements of the pieces in Rubik's Cube is

$$\frac{8!12!3^8 2^{12}}{2 \cdot 3 \cdot 2} = 2^{27}3^{14}5^3 7^2 11 = 43252003274489856000 \approx 4.3 \cdot 10^{19}. \tag{3.5}$$

This size is impressive, but its magnitude should not be construed as any kind of reason that Rubik's cube is hard to solve. After all, the letters of the alphabet can be arranged in $26! \approx 4.03 \cdot 10^{26}$ ways but it is very easy to rearrange any listing of the letters into alphabetical order. If a company came out with the Alphabet Game and said on the packaging "Over 4×10^{26} possibilities!" you would not think it must be hard since that number is so big.

The denominator $2 \cdot 3 \cdot 2 = 12$ in (3.5) comes from the three constraints in (3.4). If you were to take apart the cube and put it back together at random, it is possible you wouldn't be able to solve it. In fact, the probability is only $\frac{1}{12}$ that you can solve it, because a random choice of $(\pi, \rho, \mathbf{v}, \mathbf{w})$ will have all three conditions in (3.4) satisfied with probability $\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{12}$. You won't be able to solve it if $\operatorname{sgn} \pi \neq \operatorname{sgn} \rho$, if $\sum v_i \equiv 1, 2 \bmod 3$, or if $\sum w_j \equiv 1 \bmod 2$.

Viewing (3.1) as a direct product of four groups, (3.4) is a subgroup, since the defining conditions are preserved under componentwise operations. Is (3.4), as a subgroup of a direct product group, the group of permutations of Rubik's Cube? No. Componentwise operations in (3.1) do not match the way moves of the cube in (3.4) compose with one another. There is a different group structure on (3.1) that is needed:

$$(\pi, \rho, \mathbf{v}, \mathbf{w})(\pi', \rho', \mathbf{v}', \mathbf{w}') = (\pi\pi', \rho\rho', \mathbf{v} + \pi\mathbf{v}', \mathbf{w} + \rho\mathbf{w}'). \tag{3.6}$$

(The notation $\pi\mathbf{v}'$ means the vector in $(\mathbf{Z}/(3))^8$ obtained by permuting the 8 coordinates of $\mathbf{v}'$ according to the permutation $\pi \in S_8$. The meaning of $\rho\mathbf{w}'$ as a vector in $(\mathbf{Z}/(2))^{12}$ is similar.) The operation (3.6) is componentwise in the first two coordinates, but not in the last two coordinates. This "twisted" direct product operation is called a *semi-direct product*. The set (3.4) with the composition law (3.6) is a group, because permuting coordinates of a vector does not change the sum of the coordinates, and this is the group of movements of the pieces in Rubik's cube [1, pp. 47–48].

In addition to disassembling the cube with a screwdriver in order to solve it, you could peel off the stickers and put them back on the faces in a solved state. This is actually a really awful idea, because the adhesive holding the stickers onto the faces is seriously weakened by peeling. But let's think about the mathematical problem raised by this method: if you peel off the stickers and put them on at random, what is the probability you would be able to solve the cube? Of course the center faces must have different colors to have a chance of solving the cube. Let's agree to keep the center stickers in place; peel off only the non-center stickers and put them back on the cube at random.

The probability you can solve the cube after peeling the non-center stickers off and then randomly putting them back on is much smaller than the $\frac{1}{12}$ probability of solving the cube after taking the cube apart with a screwdriver and randomly reassembling the pieces. That is, there are far more ways to make a cube unsolvable with peeling and resticking. For instance, putting stickers with the same color on both faces of an edge piece makes the cube impossible to solve no matter what else is done with the other stickers. Other ways of making a cube unsolvable with bad color combinations on one corner piece or edge piece are easy to imagine. (Compare this with the screwdriver method of reassembly: if you place an edge into the cube in a misoriented way, this can be counterbalanced by putting in another edge in a misoriented way.)

We know the number of solvable states of the cube (with center colors fixed) is given by (3.5). The number of ways to place the 48 non-center colors onto the faces after peeling is 48!. We can't tell the difference between restickings that differ by permutations of stickers with the same color. Any particular resticking can occur in $8!^6$ ways (8! permutations of any particular color – the center colors don't change – in the same places). So the probability that peeling off the non-center faces and randomly putting them back on the cube will give you a solvable cube is

$$\frac{(8!12!3^8 2^{12}/12)(8!^6)}{48!} \approx 1.49 \cdot 10^{-14},$$

and that is far smaller than $\frac{1}{12}$.

Suppose we now allow complete freedom: even the center stickers can be removed. There are 54! ways of putting all 54 stickers back onto the cube and any particular resticking can be done in $9!^6$ ways since permuting the stickers with a fixed color doesn't change the appearance of the faces. For a resticking to be a solvable state of the cube, the center faces have to be assigned different colors. That can be done in 6! ways (no specification of which sticker of each color is actually used). If such an assignment of the center faces is made, there are $8!12!3^8 2^{12}/12$ ways to restick the remaining stickers into a solvable state of the cube. The probability that a resticking of all the stickers is a solvable state of the cube is therefore

$$\frac{(8!12!3^8 2^{12}/12)(6!)(9!^6)}{54!} \approx 3.08 \cdot 10^{-16}.$$

References

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