THE p-ADIC EXPANSION OF RATIONAL NUMBERS

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1. Introduction

In the positive real numbers, the decimal expansion of every positive rational number is eventually periodic[1] (*e.g.*, $21/55 = .3\overline{81} = .3818181\ldots$) and, conversely, every eventually periodic decimal expansion is a positive rational number. We will prove the set of all rational numbers can be characterized among the p-adic numbers a similar way: they are the p-adic numbers with eventually periodic p-adic expansions.

Example 1.1. In $\mathbf{Q}_3$
$$\frac{2}{5} = 1\overline{1210} = 1121012101210\ldots.$$
where the initial one-digit block "1" is followed by the repeating block 1210. Let's check this is correct:
$$\begin{aligned}
1\overline{1210} &= 1121012101210\ldots \\
&= 1 + 3(121012101210\ldots) \\
&= 1 + 3(1 + 2\cdot 3 + 3^2)(1 + 3^4 + 3^8 + 3^{12} + \cdots) \\
&= 1 + 3(16)\sum_{k\geq 0} 3^{4k} \\
&= 1 + \frac{48}{1-3^4} \\
&= 1 - \frac{48}{80} \\
&= \frac{32}{80} \\
&= \frac{2}{5}.
\end{aligned}$$

As above, throughout this note we will use the convention of writing p-adic expansions from left to right starting with the lowest-order term, in the same way power series are written. For example, in $\mathbf{Q}_p$ we write
$$-1 = (p-1) + (p-1)p + (p-1)p^2 + \cdots$$
rather than $-1 = \cdots + (p-1)p^2 + (p-1)p + (p-1)$. When writing positive integers in base p, we will write them from left to right starting with the highest order term, to match the way positive integers are written in base 10, and we'll include a subscript for the base. For example, 58 in base 3 is 2011_3 since $58 = 2\cdot 3^3 + 0\cdot 3^2 + 1\cdot 3 + 1$, and we'd write its 3-adic expansion as 1102 to designate $1 + 1\cdot 3 + 0\cdot 3^2 + 2\cdot 3^3$.

[1]This characterization of $\mathbf{Q}_{>0}$ inside $\mathbf{R}_{>0}$ is not affected by some numbers having more than one decimal expansion, such as $.5 = .49999\ldots$, which are both eventually periodic: eventually all 0 or eventually all 9.

Multiplying and dividing a p-adic number by powers of p shifts the digits to the left or right, but does not affect the property of having an eventually periodic p-adic expansion. Therefore it suffices to focus for the most part on numbers with p-adic absolute value 1, which are p-adic expansions of the form $c_0 + c_1 p + c_2 p^2 + \cdots$ where $0 \leq c_i \leq p-1$ and $c_0 \neq 0$.

2. Purely periodic expansions

As a warm-up, let's describe p-adic numbers with purely periodic p-adic expansions.

Theorem 2.1. *A rational number with p-adic absolute value 1 has a purely periodic p-adic expansion if and only if it lies in the real interval $[-1, 0)$.*

Proof. A purely periodic p-adic expansion having p-adic absolute value 1 with a repeating block of k digits looks like $\overline{n_0 n_1 \ldots n_{k-1}}$, where $0 \leq n_i \leq p-1$ and $n_0 \neq 0$. We can evaluate this as a fraction by summing geometric series in $\mathbf{Z}_p$:

$$\begin{aligned} \overline{n_0 n_1 \ldots n_{k-1}} &= 1(n_0 n_1 \ldots n_{k-1}) + p^k(n_0 n_1 \ldots n_{k-1}) + p^{2k}(n_0 n_1 \ldots n_{k-1}) + \cdots \\ &= (n_0 n_1 \ldots n_{k-1})(1 + p^k + p^{2k} + \cdots) \\ &= \frac{n_0 n_1 \ldots n_{k-1}}{1 - p^k}. \qquad (2.1) \end{aligned}$$

The p-adic expansion in the numerator of (2.1), which is the base p number $(n_{k-1} \cdots n_1 n_0)_p$ with digits in reverse order, is an integer between 1 and $p^k - 1$ (it is not 0 since $n_0 \neq 0$), and we are dividing it by $1 - p^k = -(p^k - 1)$, so this purely periodic expansion is a rational number lying in the interval $[-1, 0)$.

Conversely, let r be a rational number with p-adic absolute value 1 that lies in $[-1, 0)$. We will show r can be written in the form (2.1), and then the calculations that led to (2.1) can be read in reverse to see r has a purely periodic p-adic expansion.

Since $|r|_p = 1$ and $r < 0$ we can write $r = a/b$ with numerator $a < 0$ and denominator $b \geq 1$ that are both not divisible by p. Since p and b are relatively prime, from elementary number theory we have $p^k \equiv 1 \bmod b$ for some $k \geq 1$. Thus $p^k = 1 + bb'$ for some positive integer b', so

$$r = \frac{a}{b} = \frac{ab'}{bb'} = \frac{-ab'}{1 - p^k}.$$

Set $N = -ab'$. Since $a < 0$, $N \in \mathbf{Z}^+$. From $-1 \leq r < 0$ we get $-1 \leq N/(1 - p^k) < 0$, so $0 < N \leq p^k - 1$. Thus N in base p has at most k digits: $N = n_0 + n_1 p + \cdots + n_{k-1} p^{k-1}$ where the digits n_i are between 0 and $p - 1$. Hence r has the form (2.1). Since a and b' are not divisible by p, $|N|_p = 1$ so $n_0 \neq 0$. □

Remark 2.2. This theorem is not saying all rational numbers in $[-1, 0)$ have purely periodic p-adic expansions. It is the rationals in $[-1, 0)$ *with p-adic absolute value* 1 that have purely periodic expansions.

Example 2.3. Let's work out the 3-adic expansion of $-5/11$, which is in $[-1, 0)$ with 3-adic absolute value 1. The least[2] $k \geq 1$ making $3^k \equiv 1 \bmod 11$ is $k = 5$, with $3^5 - 1 = 11 \cdot 22$, so

$$-\frac{5}{11} = -\frac{5 \cdot 22}{11 \cdot 22} = -\frac{110}{3^5 - 1} = \frac{110}{1 - 3^5}.$$

[2]It is not important to pick k minimal, but to do otherwise makes the periodic digit block appear longer, like writing $\overline{12}$ as $\overline{1212}$.

In base 3, $110 = 3^4 + 3^3 + 2 = 11002_3$. Its 3-adic expansion from left to right is 20011, so

$$-\frac{5}{11} = \frac{11002_3}{1-3^5} = \frac{20011}{1-3^5} = \overline{20011} = 2001120011\ldots.$$

As a check that this calculation is correct, add up the terms in the 3-adic expansion and get back $-5/11$:

$$\begin{aligned} 2001120011\ldots &= 2\sum_{i\geq 0} 3^{5i} + 3^3\sum_{i\geq 0} 3^{5i} + 3^4\sum_{i\geq 0} 3^{5i} \\ &= \frac{2}{1-3^5} + \frac{27}{1-3^5} + \frac{81}{1-3^5} \\ &= \frac{2+27+81}{-242} \\ &= -\frac{110}{242} \\ &= -\frac{11\cdot 10}{11\cdot 22} \\ &= -\frac{5}{11}. \end{aligned}$$

We can get the p-adic expansion of a rational number in the real interval $(0,1)$ having p-adic absolute value 1 by using Theorem 2.1 to get the expansion of its negative and then negating the result. Recall the simple rule for negating a nonzero p-adic expansion: if $x = c_d p^d + c_{d+1}p^{d+1} + \cdots + c_i p^i + \cdots$ where the c_i are digits and $c_d \neq 0$, then

$$-x = (p - c_d)p^d + (p-1-c_{d+1})p^{d+1} + \cdots + (p-1-c_i)p^i + \cdots. \tag{2.2}$$

In the expansion of $-x$, note the first digit is affected differently from the rest: $p - c_d$ compared to $p-1-c_i$ for $i > d$.

Example 2.4. Let's derive the 3-adic expansion of $2/5$, which was pulled out of nowhere in Example 1.1. We will use the proof of Theorem 2.1 to find the expansion of $-2/5$ and then negate the result.

To make $3^k \equiv 1 \bmod 5$ we can use $k = 4$. Then $3^k - 1 = 5\cdot 16$, so

$$-\frac{2}{5} = -\frac{2\cdot 16}{5\cdot 16} = \frac{32}{1-3^4}.$$

In base 3, $32 = 3^3 + 3 + 2 = 1012_3$, so

$$-\frac{2}{5} = \frac{1012_3}{1-3^4} = \frac{2101}{1-3^4} = \overline{2101} = 210121012101\ldots,$$

which is purely periodic. Negating and using (2.2) with $p = 3$, we get

$$\frac{2}{5} = -210121012101\ldots = 112101210121\ldots = 1\overline{1210},$$

which is eventually periodic rather than purely periodic.

3. Eventually periodic expansions

Theorem 3.1. *In $\mathbf{Q}_p$, the numbers with eventually periodic p-adic expansions are precisely the rational numbers.*

Proof. We begin by showing every eventually periodic p-adic expansion is rational. This will generalize the calculations at the start of the proof of Theorem 2.1. An eventually periodic p-adic expansion with absolute value 1 looks like

$$m_0m_1\cdots m_{j-1}\overline{n_0n_1\cdots n_{k-1}} = m_0m_1\cdots m_{j-1}n_0n_1\cdots n_{k-1}n_0n_1\cdots n_{k-1}\ldots, \tag{3.1}$$

a first block of j digits $m_0m_1\cdots m_{j-1}$ followed by a repeating block of k digits $n_0n_1\cdots n_{k-1}$. (If the expansion is purely periodic then the initial block can be taken as empty and set $j=0$.) Write (3.1) in series form as

$$m_0+\cdots+m_{j-1}p^{j-1}+(n_0p^j+\cdots+n_{k-1}p^{j+k-1})+(n_0p^{j+k}+\cdots+n_{k-1}p^{j+2k-1})+\cdots.$$

Using geometric series, we evaluate (3.1):

$$\begin{aligned} m_0\ldots m_{j-1}\overline{n_0\ldots n_{k-1}} &= m_0\ldots m_{j-1}+(n_0\ldots n_{k-1})(p^j+p^{j+k}+p^{j+2k}+\cdots) \\ &= m_0\ldots m_{j-1}+p^j(n_0\ldots n_{k-1})(1+p^k+p^{2k}+\cdots) \\ &= m_0\ldots m_{j-1}+p^j\frac{n_0\ldots n_{k-1}}{1-p^k} \\ &= (m_{j-1}\ldots m_0)_p+p^j\frac{(n_{k-1}\ldots n_0)_p}{1-p^k}, \end{aligned}$$

which is a rational number. (This generalizes the calculations that led to (2.1), which is the special case $j=0$.) Allowing multiplication or division by powers of p, we have shown all eventually periodic p-adic expansions are rational numbers.

To prove the converse, that every rational number r has an eventually periodic p-adic expansion, we will, perhaps surprisingly, focus on *negative* r. The p-adic expansion of a positive rational number can be obtained from its negative by negating with (2.2), which clearly shows the negation of an eventually periodic p-adic expansion is eventually periodic. (If $r\in\mathbf{Z}^+$ there's really no need to negate first: the base p expansion of r is its p-adic expansion.)

Case 1: $r\in\mathbf{Z}$ with $r<0$. Write $r=-R$ with $R\in\mathbf{Z}^+$. There is a $j\geq 1$ such that $R<p^j$. Then

$$r=-R=(p^j-R)-p^j.$$

Since p^j-R is an integer in $\{1,\ldots,p^j-1\}$ we can write it in base p as $c_0+\cdots+c_{j-1}p^{j-1}$. Then

$$r=(p^j-R)-p^j=\sum_{i=0}^{j-1}c_ip^i+\sum_{i\geq j}(p-1)p^i,$$

which is eventually periodic since its digits eventually all equal $p-1$.

Case 2: $r\in\mathbf{Q}\cap\mathbf{Z}_p^\times\cap(-1,0)$. The p-adic expansion of r is purely periodic by Theorem 2.1, and the proof of that theorem shows how to obtain the expansion.

Case 3: $r\in\mathbf{Q}\cap\mathbf{Z}_p\cap(-1,0)$. Write $r=p^nu$ with $u\in\mathbf{Z}_p^\times$. Then $u=r/p^n$ is rational, of p-adic absolute value 1, and is in the interval $(-1/p^n,0)\subset(-1,0)$, so u has a purely periodic p-adic expanion by Case 2. Therefore $r=p^nu$ has the same purely periodic expansion except for starting n positions further to the right.

Case 4: $r\in\mathbf{Q}\cap\mathbf{Z}_p$, $r\notin\mathbf{Z}$, and $r<-1$. The number r lies strictly between two negative integers: $-(N+1)<r<-N$ for some positive integer N, so $-1<r+N<0$. Since $r+N\in\mathbf{Z}_p$, by Case 3 the p-adic expansion of $r+N$ is purely periodic, although not necessarily starting at the p^0-digit (since $r+N$ might not be in $\mathbf{Z}_p^\times$), so we can write

$$r+N=\sum_{i\geq 0}a_ip^i \tag{3.2}$$

where $a_i\in\{0,1,\ldots,p-1\}$ and the a_i are purely periodic after a possible initial string of zero digits. Since $r+N$ is not a positive integer, the p-adic expansion (3.2) has infinitely many nonzero a_i. Thus the partial sums $a_0+a_1p+\cdots+a_{j-1}p^{j-1}$ become arbitrarily large in the usual sense as j grows, so there is a j such that

$$a_0+a_1p+\cdots+a_{j-1}p^{j-1}>N. \tag{3.3}$$

Let j be the smallest choice fitting this inequality, so $a_{j-1}\neq 0$. Then

$$r+N=(a_0+a_1p+\cdots+a_{j-1}p^{j-1})+\sum_{i\geq j}a_ip^i$$

so

$$r=(a_0+a_1p+\cdots+a_{j-1}p^{j-1}-N)+\sum_{i\geq j}a_ip^i \tag{3.4}$$

and the difference $a_0+a_1p+\cdots+a_{j-1}p^{j-1}-N$ is a positive integer by (3.3) that is less than $(p-1)+(p-1)p+\cdots+(p-1)p^{j-1}=p^j-1$, so we can write the difference in base p:

$$a_0+a_1p+\cdots+a_{j-1}p^{j-1}-N=a_0'+a_1'p+\cdots+a_{j-1}'p^{j-1}$$

with $0\leq a_i'\leq p-1$, so (3.4) becomes

$$r=(a_0'+a_1'p+\cdots+a_{j-1}'p^{j-1})+\sum_{i\geq j}a_ip^i.$$

This is an eventually periodic p-adic expansion since the a_i for $i\geq j$ are eventually periodic.

Case 5: $r\in\mathbf{Q}$, $r\notin\mathbf{Z}_p$, $r<0$. Since $p^er\in\mathbf{Z}_p$ for large e, we can use a previous case on p^er and then divide by p^e. □

4. Examples

The proof of Theorem 3.1 gives an algorithm to compute the p-adic expansion of any rational number in $\mathbf{Z}_p$:

(1) Assume $r<0$. (If $r>0$, apply the rest of the algorithm to $-r$ and then negate with (2.2) to get the expansion for r.)

(2) If $r\in\mathbf{Z}_{<0}$ then write $r=-R$ and pick $j\geq 1$ such that $R<p^j$. Then $r=(p^j-R)-p^j=(p^j-R)+\sum_{i\geq j}(p-1)p^i$ and p^j-R has a base p expansion not going beyond the p^{j-1}-digit.

(3) If $-1<r<0$ let $r=p^nu$ with $u\in\mathbf{Z}_p^\times$. Then $u\in(-1,0)$ and the p-adic expanion of u is purely periodic using the proof of Theorem 2.1. Multiplying it by p^n gives the (purely periodic) p-adic expansion of r.

(4) If $-(N+1)<r<-N$ for an integer $N\geq 1$ then $-1<r+N<0$, so the expansion of $r+N$ is obtained by the previous step, say $r+N=\sum_{i\geq 0}a_ip^i$. Pick the first truncation $a_0+a_1p+\cdots+a_{j-1}p^{j-1}$ in this expansion that exceeds N, so $r=(\sum_{i=0}^{j-1}a_ip^i-N)+\sum_{i\geq j}a_ip^i$. The difference in parentheses is a positive integer and its base p expansion has the form $\sum_{i=0}^{j-1}a_i'p^i$, so $r=\sum_{i=0}^{j-1}a_i'p^i+\sum_{i\geq j}a_ip^i$.

Example 4.1. Let's work out the p-adic expansion of $77/18$ in $\mathbf{Q}_2$, $\mathbf{Q}_3$, $\mathbf{Q}_5$, and $\mathbf{Q}_7$.

Expansion of $77/18$ in $\mathbf{Q}_2$: Since $77/18 = (1/2)(77/9)$ and $|77/9|_2 = 1$, we will get the 2-adic expansion of $77/9$ and then divide through by 2. And since $77/9 > 0$, we will first get the 2-adic expansion of $-77/9$ and then negate what we find.

Let $r = -77/9$. Since $-9 < r < -8$, set $N = 8$. Since $-1 < r + 8 < 0$ and $r + 8 = -5/9 \in \mathbf{Z}_2^\times \cap (-1, 0)$ we will find the 2-adic expansion of $-5/9$ by Theorem 2.1. The least k making $2^k \equiv 1 \bmod 9$ is $k = 6$:

$$2^6 - 1 = 63 = 9 \cdot 7 \Longrightarrow -\frac{5}{9} = -\frac{5 \cdot 7}{63} = \frac{35}{1 - 2^6}.$$

In base 2, $35 = 1 + 2 + 2^5 = 100011_2$, so

$$\frac{35}{1 - 2^6} = \frac{100011_2}{1 - 2^6} = \frac{110001}{1 - 2^6} = \overline{110001} = 110001110001110001\ldots$$

The first truncation of this that exceeds $N = 8$ is $110001 = 35$, so

$$r = -8 - \frac{5}{9} = -8 + 110001 + 000000\overline{110001} = (35 - 8) + 000000\overline{110001}.$$

Since $35 - 8 = 27 = 11011_2$, which has 2-adic expansion 11011 (it is palindromic, a coincidence), we get

$$r = -\frac{77}{9} = 11011 + 000000\overline{110001} = 110110\overline{110001}.$$

Thus

$$\frac{77}{9} = -110110\overline{110001} = 101001\overline{001110},$$

so

$$\frac{77}{18} = \frac{101001\overline{001110}}{2} = \frac{1}{2} + 01001\overline{001110}.$$

Let's check: in $\mathbf{Q}_2$,

$$\frac{1}{2} + 01001\overline{001110} = \frac{1}{2} + (2 + 16) + 2^5\frac{4 + 8 + 16}{1 - 2^6} = \frac{1}{2} + 18 + 32\frac{28}{1 - 64} = \frac{37}{2} - \frac{32 \cdot 4}{9} \stackrel{\checkmark}{=} \frac{77}{18}.$$

Expansion of $77/18$ in $\mathbf{Q}_3$: Since $77/18 = (1/9)(77/2)$, first we will figure out the 3-adic expansion of $77/2$ and then divide it by 9. Since $77/2 > 0$, first we will compute the 3-adic expansion of $-77/2$ and then negate.

Let $r = -77/2$, so $-39 < r < -38$. We have $r + 38 = -1/2$, which is easy to expand 3-adically:

$$-\frac{1}{2} = \frac{1}{1 - 3} = \overline{1} = 111\ldots$$

and the first truncation of this expansion that exceeds 38 is $1111 = 40$, so

$$r = -38 - \frac{1}{2} = -38 + 1111 + 0000\overline{1} = (40 - 38) + 0000\overline{1} = 2000\overline{1}.$$

Therefore

$$\frac{77}{2} = -2000\overline{1} = 1222\overline{1}$$

so

$$\frac{77}{18} = \frac{1222\overline{1}}{9} = \frac{1}{9} + \frac{2}{3} + 22\overline{1}.$$

Let's check: in $\mathbf{Q}_3$,

$$\frac{1}{9} + \frac{2}{3} + 22\overline{1} = \frac{1}{9} + \frac{2}{3} + (2 + 2 \cdot 3) + \frac{9}{1 - 3} = \frac{7}{9} + 8 - \frac{9}{2} = \frac{14 + 18 \cdot 8 - 81}{18} \stackrel{\checkmark}{=} \frac{77}{18}.$$

Expansion of 77/18 in $\mathbf{Q}_5$: We'll get the expansion for $-77/18$ and then negate.

Let $r = -77/18$. Since $-5 < r < -4$, set $N = 4$. Then $-1 < r + 4 < 0$ and $r + 4 = -5/18 = 5(-1/18) = 5u$ where $u = -1/18 \in \mathbf{Z}_5^\times \cap (-1, 0)$. We will get the 5-adic expansion of $-1/18$ using Theorem 2.1 and then multiply through by 5.

The least k making $5^k \equiv 1 \bmod 18$ is $k = 6$:

$$5^6 - 1 = 15624 = 18 \cdot 868 \Longrightarrow -\frac{1}{18} = -\frac{868}{15624} = \frac{868}{1-5^6}.$$

In base 5, $868 = 3 + 3 \cdot 5 + 4 \cdot 5^2 + 5^3 + 5^4 = 11433_5$, so

$$u = \frac{868}{1-5^6} = \frac{11433_5}{1-5^6} = \frac{33411}{1-5^6} = \overline{334110} = 334110334110033411\ldots$$

Thus

$$-\frac{5}{18} = 5u = \overline{033411}.$$

The first truncation of this that exceeds $N = 4$ is 03, which is 15, so

$$r = -4 - \frac{5}{18} = -4 + 03 + 00\overline{341103} = (15-4) + 00\overline{34110}.$$

Since $15 - 4 = 11 = 21_5$, which has 5-adic expansion 12,

$$r = -\frac{77}{18} = 12 + 00\overline{341103} = 12\overline{341103}.$$

Thus

$$\frac{77}{18} = -12\overline{341103} = 42\overline{103341}.$$

Let's check: in $\mathbf{Q}_5$,

$$42\overline{103341} = 4 + 2 \cdot 5 + 5^2 \frac{1 + 3 \cdot 5^2 + 3 \cdot 5^3 + 4 \cdot 5^4 + 5^5}{1-5^6} = 14 + 25\frac{6076}{1-5^6} = 14 - 25\frac{7}{18} \overset{\checkmark}{=} \frac{77}{18}.$$

Expansion of 77/18 in $\mathbf{Q}_7$: We'll get the expansion for $-11/18$ and then multiply by -7.

Let $r = -11/18$. It lies in $\mathbf{Z}_7^\times \cap (-1, 0)$ so we can compute its 7-adic expansion from Theorem 2.1.

The least k making $7^k \equiv 1 \bmod 18$ is $k = 3$:

$$7^3 - 1 = 342 = 18 \cdot 19 \Longrightarrow -\frac{11}{18} = -\frac{11 \cdot 19}{342} = \frac{209}{1-7^3}.$$

In base 7, $209 = 6 + 7 + 4 \cdot 7^2 = 416_7$, so

$$r = \frac{209}{1-7^3} = \frac{416_7}{1-7^3} = \frac{614}{1-7^3} = \overline{614} = 614614614\ldots$$

Therefore

$$\frac{11}{18} = -614614614\ldots = 152052052\ldots = 1\overline{520}$$

so

$$\frac{77}{18} = 7\left(\frac{11}{18}\right) = 01\overline{520}.$$

Let's make our final check: in $\mathbf{Q}_7$,

$$01\overline{520} = 7 + 7^2\frac{5 + 2 \cdot 7}{1-7^3} = 7 - 49\frac{19}{342} = 7 - \frac{49}{18} \overset{\checkmark}{=} \frac{77}{18}.$$