LOCAL COMPACTNESS OF THE DUAL GROUP USING ASCOLI

Let G be a locally compact abelian group and $\widehat{G}$ be its dual group, which is also abelian. We will explain how to make $\widehat{G}$ into a locally compact group using the compact-open topology.

Theorem 1. *If G is a locally compact abelian group, then $\widehat{G}$ is a topological group in the compact-open topology.*

Proof. For $\chi \in \widehat{G}$, the basic open sets around χ in the compact-open topology on $\widehat{G}$ are of the form $N_\chi(K,\varepsilon) = \{\psi \in \widehat{G} : |\psi(x) - \chi(x)| < \varepsilon \text{ for all } x \in K\}$, where K is a nonempty compact subset of G and $\varepsilon > 0$. We need to show multiplication $m\colon \widehat{G} \times \widehat{G} \to \widehat{G}$ and inversion $\widehat{G} \to \widehat{G}$ are continuous for the compact-open topology on $\widehat{G}$.

To show multiplication is continuous, pick characters χ and χ' in $\widehat{G}$, a nonempty compact subset K in G, and $\varepsilon > 0$. Then $N_\chi(K,\varepsilon/2) \times N_{\chi'}(K,\varepsilon/2)$ is an open set around (χ,χ') in $\widehat{G} \times \widehat{G}$ that m maps into $N_{\chi\chi'}(K,\varepsilon)$ by the triangle inequality: if $\psi \in N_\chi(K,\varepsilon/2)$ and $\psi' \in N_{\chi'}(K,\varepsilon/2)$, then for all $x \in K$ we have

$$\begin{aligned} |\psi(x)\psi'(x) - \chi(x)\chi'(x)| &= |(\psi(x) - \chi(x))\psi'(x) + (\psi'(x) - \chi'(x))\chi(x)| \\ &\leq |\psi(x) - \chi(x)| + |\psi'(x) - \chi'(x)| \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon, \end{aligned}$$

so $\psi\psi' \in N_{\chi\chi'}(K,\varepsilon)$.

To show inversion is continuous, pick $\chi \in \widehat{G}$, a nonempty compact subset K of G, and $\varepsilon > 0$. Then $N_\chi(K,\varepsilon)$ is an open set containing χ and inversion maps it into $N_{\chi^{-1}}(K,\varepsilon)$:

$$\psi \in N_\chi(K,\varepsilon),\ x \in K \Longrightarrow |\psi^{-1}(x) - \chi^{-1}(x)| = \left|\frac{\chi(x) - \psi(x)}{\psi(x)\chi(x)}\right| = |\chi(x) - \psi(x)| < \varepsilon,$$

so $\psi^{-1} \in N_{\chi^{-1}}(K,\varepsilon)$. $\square$

To show $\widehat{G}$ is locally compact in the compact-open topology, the usual proof proceeds through Banach algebras, Alaoglu's theorem, L^1-L^∞ duality, and a comparison between the compact-open topology and the topology of pointwise convergence on $\widehat{G}$. We will give a proof in Theorem 5 that $\widehat{G}$ is locally compact in the compact-open topology using no additional topologies, no Banach algebras, *etc.* Our main tool will be the standard theorem describing when sets of functions are compact: Ascoli's theorem.

Lemma 2. *Fix $f \in L^1(G)$. For $y \in G$, let $L_y f\colon G \to \mathbf{C}$ by $(L_y f)(x) = f(yx)$. The map $y \mapsto L_y f$ from G to $L^1(G)$ is continuous.*

The lemma is proved by checking it directly on the dense subset $C_c(G) \subset L^1(G)$ and then extending it to all of $L^1(G)$ by an approximation argument. Details are left to the reader.

Next we prove a result about the decay of Fourier transforms. For $f \in L^1(G)$, its Fourier transform is $\widehat{f}(\chi) = \int_G f(x)\overline{\chi}(x)\,\mathrm{d}x$, where $\mathrm{d}x$ is some choice of left Haar measure on G. Using the compact-open topology on $\widehat{G}$, the function $\widehat{f}\colon \widehat{G} \to \mathbf{C}$ is (uniformly) continuous.

Theorem 3. *If $f \in L^1(G)$ then $\widehat{f}\colon \widehat{G} \to \mathbf{C}$ "vanishes at ∞": for any $\varepsilon > 0$ there is a compact set $C \subset \widehat{G}$ such that $|\widehat{f}(\chi)| < \varepsilon$ for all $\chi \notin C$.*

Proof. Since $\widehat{f}\colon \widehat{G} \to \mathbf{C}$ is continuous when $\widehat{G}$ has the compact-open topology, our task would follow from showing for any $\varepsilon > 0$ that the (closed) set

$$C := \{\chi \in \widehat{G} : |\widehat{f}(\chi)| \geq \varepsilon\}$$

is compact in $\widehat{G}$ using the compact-open topology.

Since $\widehat{G}$ is a closed subset of the space $C(G, S^1)$ of continuous functions from G to S^1, what we need to do is show the above set is compact in $C(G, S^1)$. For this, Ascoli's theorem tells us exactly what has to be checked: equicontinuity of the characters in C at each point of G. Since we're dealing with characters and the compact-open topology, it is enough to check equicontinuity of the characters in C at the identity e of G. So for each $\delta > 0$ we want to find an open neighborhood $U = U_\delta$ of e such that

$$y \in U, \quad |\widehat{f}(\chi)| \geq \varepsilon \Longrightarrow |\chi(y) - 1| < \delta.$$

It's not evident how to turn a lower bound on the Fourier transform at χ into an upper bound on $\chi(y) - 1$. The trick is to get a bound on $|\chi(y) - 1|$ where y doesn't show up in $\chi(y)$ anymore.

For any $\chi \in \widehat{G}$ such that $|\widehat{f}(\chi)| \geq \varepsilon$ and any $y \in G$, we have

$$\begin{aligned}
\varepsilon|\chi(y) - 1| &\leq |(\overline{\chi}(y) - 1)\widehat{f}(\chi)| \\
&= \left|(\overline{\chi}(y) - 1)\int_G f(x)\overline{\chi}(x)\,\mathrm{d}x\right| \\
&= \left|\int_G f(x)\overline{\chi}(xy)\,\mathrm{d}x - \int_G f(x)\overline{\chi}(x)\,\mathrm{d}x\right| \\
&= \left|\int_G f(xy^{-1})\overline{\chi}(x)\,\mathrm{d}x - \int_G f(x)\overline{\chi}(x)\,\mathrm{d}x\right| \\
&= \left|\int_G (f(xy^{-1}) - f(x))\overline{\chi}(x)\,\mathrm{d}x\right| \\
&\leq \int_G \left|f(xy^{-1}) - f(x)\right|\,\mathrm{d}x \\
&= |L_{y^{-1}}f - f|_1,
\end{aligned}$$

so

$$|\chi(y) - 1| \leq \frac{1}{\varepsilon}|L_{y^{-1}}f - f|_1.$$

From continuity of $y \mapsto L_y f$ and continuity of inversion on G, $|L_{y^{-1}}f - f|_1 \to 0$ as $y \to e$ in G. Therefore $|\chi(y) - 1| < \delta$ for all y near e, and that level of nearness to e gives us the desired set U. □

Remark 4. Theorem 3 is a generalization of the Riemann–Lebesgue lemma, which is the special case $G = S^1 = \mathbf{R}/2\pi\mathbf{Z}$: if $f\colon \mathbf{R} \to \mathbf{C}$ is 2π-periodic and integrable then its Fourier coefficients $\widehat{f}(n) = \int_0^{2\pi} f(x)e^{-2\pi inx}\,\mathrm{d}x$ tend to 0 as $|n| \to \infty$.

Theorem 5. *When G is a locally compact abelian group, the group $\widehat{G}$ is locally compact in the compact-open topology.*

Proof. Since $\widehat{G}$ is a topological group, it suffices to show there is a neighborhood basis of the trivial character $\mathbf{1}$ consisting of open subsets with compact closures. Every neighborhood of $\mathbf{1}$ in $\widehat{G}$ contains some $N_{\mathbf{1}}(K,\varepsilon) = \{\chi \in \widehat{G} : |\chi(x) - 1| < \varepsilon \text{ for all } x \in K\}$, where K is a nonempty compact subset of G and $\varepsilon > 0$.

When $0 < \varepsilon < 1$ and K has positive Haar measure, we will show $N_{\mathbf{1}}(K,\varepsilon)$ has compact closure in $\widehat{G}$ by copying the argument from [1, p. 362]. Let $f = \xi_K$ be the characteristic function of K, so $f \in L^1(G)$ because compact subsets of G have finite Haar measure. Let $\mu(K)$ be the Haar measure of K. For any $\chi \in \widehat{G}$,

$$\mu(K) = \int_G \xi_K \, \mathrm{d}x = \int_G \xi_K \cdot (1 - \chi) \, \mathrm{d}x + \int_G \xi_K \cdot \chi \, \mathrm{d}x.$$

Taking absolute values, if $\chi \in N_{\mathbf{1}}(K,\varepsilon)$ then $\mu(K) \le \mu(K)\varepsilon + |\widehat{\xi}_K(\chi)|$, so

$$|\widehat{\xi}_K(\chi)| \ge (1 - \varepsilon)\mu(K) > 0.$$

By Theorem 3, the set of χ fitting this inequality is a compact set, so $N_{\mathbf{1}}(K,\varepsilon)$ has compact closure in $\widehat{G}$.

What if $\varepsilon \ge 1$ or K has measure 0? We want to reduce these cases to the previous case by passing to a smaller neighborhood where both $\varepsilon < 1$ and K has positive Haar measure. Making ε smaller makes $N_{\mathbf{1}}(K,\varepsilon)$ smaller, so by passing to a suitable smaller neighborhood of $\mathbf{1}$ we can assume $\varepsilon < 1$. Making K larger makes $N_{\mathbf{1}}(K,\varepsilon)$ smaller, so we can reduce to the case that K has positive measure if we show any compact subset of G with measure 0 is contained in a compact subset of G with positive measure. Since G itself has positive Haar measure, by inner regularity of Haar measure there's a compact subset L of G such that L has positive measure. Then $K \cup L$ is compact with positive measure and contains K. □

References

[1] E. Hewitt and K. A. Ross, "Abstract Harmonic Analysis, I", Springer-Verlag, New York, 1963.