Let $G$ be a locally compact abelian group and $\hat{G}$ be its dual group, which is also abelian. We will explain how to make $\hat{G}$ into a locally compact group using the compact-open topology.

**Theorem 1.** If $G$ is a locally compact abelian group, then $\hat{G}$ is a topological group in the compact-open topology.

**Proof.** For $\chi \in \hat{G}$, the basic open sets around $\chi$ in the compact-open topology on $\hat{G}$ are of the form $N_\chi(K,\varepsilon) = \{\psi \in \hat{G} : |\psi(x) - \chi(x)| < \varepsilon \text{ for all } x \in K\}$, where $K$ is a nonempty compact subset of $G$ and $\varepsilon > 0$. We need to show multiplication $m: \hat{G} \times \hat{G} \rightarrow \hat{G}$ and inversion $\hat{\cdot}: \hat{G} \rightarrow \hat{G}$ are continuous for the compact-open topology on $\hat{G}$.

To show multiplication is continuous, pick characters $\chi$ and $\chi'$ in $\hat{G}$, a nonempty compact subset $K$ in $G$, and $\varepsilon > 0$. Then $N_\chi(K,\varepsilon/2) \times N_{\chi'}(K,\varepsilon/2)$ is an open set around $(\chi, \chi')$ in $\hat{G} \times \hat{G}$ that $m$ maps into $N_{\chi\chi'}(K,\varepsilon)$ by the triangle inequality: if $\psi \in N_\chi(K,\varepsilon/2)$ and $\psi' \in N_{\chi'}(K,\varepsilon/2)$, then for all $x \in K$ we have

$$|\psi(x)\psi'(x) - \chi(x)\chi'(x)| = |(\psi(x) - \chi(x))\psi'(x) + (\psi'(x) - \chi'(x))\chi(x)| \\
\leq |\psi(x) - \chi(x)| + |\psi'(x) - \chi'(x)| \\
< \varepsilon/2 + \varepsilon/2 \\
= \varepsilon,$$

so $\psi\psi' \in N_{\chi\chi'}(K,\varepsilon)$.

To show inversion is continuous, pick $\chi \in \hat{G}$, a nonempty compact subset $K$ of $G$, and $\varepsilon > 0$. Then $N_\chi(K,\varepsilon)$ is an open set containing $\chi$ and inversion maps it into $N_{\chi^{-1}}(K,\varepsilon)$:

$$\psi \in N_\chi(K,\varepsilon), \ x \in K \implies |\psi^{-1}(x) - \chi^{-1}(x)| = \left|\frac{\chi(x) - \psi(x)}{\psi(x)\chi(x)}\right| = |\chi(x) - \psi(x)| < \varepsilon,$$

so $\psi^{-1} \in N_{\chi^{-1}}(K,\varepsilon)$.

To show $\hat{G}$ is locally compact in the compact-open topology, the usual proof proceeds through Banach algebras, Alaoglu’s theorem, $L^1$-$L^\infty$ duality, and a comparison between the compact-open topology and the topology of pointwise convergence on $\hat{G}$. We will give a proof in Theorem 5 that $\hat{G}$ is locally compact in the compact-open topology using no additional topologies, no Banach algebras, etc. Our main tool will be the standard theorem describing when sets of functions are compact: Ascoli’s theorem.

**Lemma 2.** Fix $f \in L^1(G)$. For $y \in G$, let $L_yf: G \rightarrow C$ by $(L_yf)(x) = f(yx)$. The map $y \mapsto L_yf$ from $G$ to $L^1(G)$ is continuous.

The lemma is proved by checking it directly on the dense subset $C_c(G) \subset L^1(G)$ and then extending it to all of $L^1(G)$ by an approximation argument. Details are left to the reader.

Next we prove a result about the decay of Fourier transforms. For $f \in L^1(G)$, its Fourier transform is $\hat{f}(\chi) = \int_G f(x)\overline{\chi}(x) \, dx$, where $dx$ is some choice of left Haar measure on $G$. Using the compact-open topology on $\hat{G}$, the function $\hat{f}: \hat{G} \rightarrow C$ is (uniformly) continuous.
Theorem 3. If \( f \in L^1(G) \) then \( \hat{f}: \hat{G} \to \mathbb{C} \) “vanishes at \( \infty \)”: for any \( \varepsilon > 0 \) there is a compact set \( C \subset \hat{G} \) such that \( |\hat{f}(\chi)| < \varepsilon \) for all \( \chi \notin C \).

Proof. Since \( \hat{f}: \hat{G} \to \mathbb{C} \) is continuous when \( \hat{G} \) has the compact-open topology, our task would follow from showing for any \( \varepsilon > 0 \) that the (closed) set

\[
C := \{ \chi \in \hat{G} : |\hat{f}(\chi)| \geq \varepsilon \}
\]

is compact in \( \hat{G} \) using the compact-open topology.

Since \( \hat{G} \) is a closed subset of the space \( C(G,S^1) \) of continuous functions from \( G \) to \( S^1 \), what we need to do is show the above set is compact in \( C(G,S^1) \). For this, Ascoli’s theorem tells us exactly what has to be checked: equicontinuity of the characters in \( C \) at each point of \( G \). Since we’re dealing with characters and the compact-open topology, it is enough to check equicontinuity of the characters in \( C \) at the identity \( e \) of \( G \). So for each \( \delta > 0 \) we want to find an open neighborhood \( U = U_\delta \) of \( e \) such that

\[
y \in U, \quad |\hat{f}(\chi)| \geq \varepsilon \implies |\chi(y) - 1| < \delta.
\]

It’s not evident how to turn a lower bound on the Fourier transform at \( \chi \) into an upper bound on \( |\chi(y) - 1| \). The trick is to get a bound on \( |\chi(y) - 1| \) where \( y \) doesn’t show up in \( \chi(y) \) anymore.

For any \( \chi \in \hat{G} \) such that \( |\hat{f}(\chi)| \geq \varepsilon \) and any \( y \in G \), we have

\[
\varepsilon |\chi(y) - 1| \leq |(\chi(y) - 1)\hat{f}(\chi)|
\]

\[
= |(\chi(y) - 1) \int_G f(x)\overline{\chi(x)} \, dx|
\]

\[
= \left| \int_G f(x)\overline{\chi(xy)} \, dx - \int_G f(x)\overline{\chi(x)} \, dx \right|
\]

\[
= \left| \int_G f(xy^{-1})\overline{\chi(x)} \, dx - \int_G f(x)\overline{\chi(x)} \, dx \right|
\]

\[
= \left| \int_G (f(xy^{-1}) - f(x))\overline{\chi(x)} \, dx \right|
\]

\[
\leq \int_G |f(xy^{-1}) - f(x)| \, d|x|
\]

\[
= |L_{y^{-1}}f - f|_1,
\]

so

\[
|\chi(y) - 1| \leq \frac{1}{\varepsilon}|L_{y^{-1}}f - f|_1.
\]

From continuity of \( y \mapsto L_yf \) and continuity of inversion on \( G \), \( |L_{y^{-1}}f - f|_1 \to 0 \) as \( y \to e \) in \( G \). Therefore \( |\chi(y) - 1| < \delta \) for all \( y \) near \( e \), and that level of nearness to \( e \) gives us the desired set \( U \). \( \square \)

Remark 4. Theorem 3 is a generalization of the Riemann–Lebesgue lemma, which is the special case \( G = S^1 = \mathbb{R}/2\pi\mathbb{Z} \): if \( f: \mathbb{R} \to \mathbb{C} \) is 2\( \pi \)-periodic and integrable then its Fourier coefficients \( \hat{f}(n) = \int_0^{2\pi} f(x)e^{-2\pi inx} \, dx \) tend to 0 as \( |n| \to \infty \).

Theorem 5. When \( G \) is a locally compact abelian group, the group \( \hat{G} \) is locally compact in the compact-open topology.
Proof. Since $\hat{G}$ is a topological group, it suffices to show there is a neighborhood basis of the trivial character $1$ consisting of open subsets with compact closures. Every neighborhood of $1$ in $\hat{G}$ contains some $N_1(K, \varepsilon) = \{ \chi \in \hat{G} : |\chi(x) - 1| < \varepsilon \text{ for all } x \in K \}$, where $K$ is a nonempty compact subset of $G$ and $\varepsilon > 0$.

When $0 < \varepsilon < 1$ and $K$ has positive Haar measure, we will show $N_1(K, \varepsilon)$ has compact closure in $\hat{G}$ by copying the argument from [1, p. 362]. Let $f = \xi_K$ be the characteristic function of $K$, so $f \in L^1(G)$ because compact subsets of $G$ have finite Haar measure. Let $\mu(K)$ be the Haar measure of $K$. For any $\chi \in \hat{G}$,

$$\mu(K) = \int_G \xi_K \, dx = \int_G \xi_K \cdot (1 - \chi) \, dx + \int_G \xi_K \cdot \chi \, dx.$$  

Taking absolute values, if $\chi \in N_1(K, \varepsilon)$ then $\mu(K) \leq \mu(K)\varepsilon + |\hat{\xi}_K(\chi)|$, so

$$|\hat{\xi}_K(\chi)| \geq (1 - \varepsilon)\mu(K) > 0.$$  

By Theorem 3, the set of $\chi$ fitting this inequality is a compact set, so $N_1(K, \varepsilon)$ has compact closure in $\hat{G}$.

What if $\varepsilon \geq 1$ or $K$ has measure 0? We want to reduce these cases to the previous case by passing to a smaller neighborhood where both $\varepsilon < 1$ and $K$ has positive Haar measure. Making $\varepsilon$ smaller makes $N_1(K, \varepsilon)$ smaller, so by passing to a suitable smaller neighborhood of $1$ we can assume $\varepsilon < 1$. Making $K$ larger makes $N_1(K, \varepsilon)$ smaller, so we can reduce to the case that $K$ has positive measure if we show any compact subset of $G$ with measure 0 is contained in a compact subset of $G$ with positive measure. Since $G$ itself has positive Haar measure, by inner regularity of Haar measure there’s a compact subset $L$ of $G$ such that $L$ has positive measure. Then $K \cup L$ is compact with positive measure and contains $K$. □

References