

BINOMIAL COEFFICIENTS AND p -ADIC LIMITS

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Here are the power series for $\sqrt{1+x}$, $\sqrt[3]{1+x}$, and $\sqrt[6]{1+x}$:

$$\begin{aligned}\sqrt{1+x} &= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \frac{7}{256}x^5 - \frac{21}{1024}x^6 + \cdots, \\ \sqrt[3]{1+x} &= 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 - \frac{10}{243}x^4 + \frac{22}{729}x^5 - \frac{154}{6561}x^6 + \cdots, \\ \sqrt[6]{1+x} &= 1 + \frac{1}{6}x - \frac{5}{72}x^2 + \frac{55}{1296}x^3 - \frac{935}{31104}x^4 + \frac{4301}{186624}x^5 - \frac{124729}{6718464}x^6 + \cdots.\end{aligned}$$

The coefficients are all rational and are written in reduced form. It is not hard to get a formula for the coefficient of x^k in $\sqrt[n]{1+x} = (1+x)^{1/n}$:

$$\frac{d^k}{dx^k}(1+x)^{1/n} = \frac{1}{n} \left(\frac{1}{n} - 1\right) \left(\frac{1}{n} - 2\right) \cdots \left(\frac{1}{n} - (k-1)\right) (1+x)^{1/n-k}$$

so

$$\left. \frac{(\sqrt[n]{1+x})^{(k)}}{k!} \right|_{x=0} = \frac{(1/n)(1/n-1)\cdots(1/n-(k-1))}{k!} = \binom{1/n}{k}.$$

These numbers are binomial coefficients evaluated at $1/n$.

Look at the primes in the denominators of the coefficients. In $\sqrt{1+x}$ each coefficient has denominator that is a power of 2, in $\sqrt[3]{1+x}$ each coefficient has denominator that is a power of 3 ($243 = 3^5$ and $729 = 3^6$), and in $\sqrt[6]{1+x}$ each coefficient is a power of 2 times a power of 3 ($1296 = 2^4 3^4$, $31104 = 2^7 3^5$, $186624 = 2^8 3^6$, and $6718464 = 2^{10} 3^8$).

For $r \in \mathbf{Q}$ the power series for $(1+x)^r$ at $x=0$ has coefficients that are binomial coefficients evaluated at r :

$$(1+x)^r = \sum_{k \geq 0} \frac{r(r-1)(r-2)\cdots(r-(k-1))}{k!} x^k = \sum_{k \geq 0} \binom{r}{k} x^k.$$

What primes occur in the denominators of $\binom{r}{k}$?

Theorem 1. *For a rational number r , a prime dividing the denominator of $\binom{r}{k}$ must divide the denominator of r .*

Proof. We want to show if a prime p divides the denominator of $\binom{r}{k}$ then it divides the denominator of r . Writing $r = a/b$ in reduced form,

$$\binom{a/b}{k} = \frac{(a/b)(a/b-1)(a/b-2)\cdots(a/b-(k-1))}{k!} = \frac{a(a-b)(a-2b)\cdots(a-kb)}{b^k k!}$$

and it is obvious that the only possible primes in the denominator are prime factors of b or a prime factor of $k!$. It is not obvious why every prime factor of $k!$ that is not a factor of b gets completely canceled out when the ratio in $\binom{r}{k}$ is simplified. We will explain this purely algebraic phenomenon by using p -adic limits.

To prove the theorem we will prove its contrapositive: each prime p that does *not* divide the denominator of r also does not divide the denominator of any $\binom{r}{k}$. Expressed in terms of p -adic absolute values, this says: if $|r|_p \leq 1$ then $|\binom{r}{k}|_p \leq 1$ for $k \geq 0$. To prove this, observe that the expression

$$\binom{x}{k} = \frac{x(x-1)(x-2)\cdots(x-(k-1))}{k!}$$

is a polynomial with rational coefficients, and therefore it is a *continuous function* $\mathbf{Q}_p \rightarrow \mathbf{Q}_p$ just as much as it is a continuous function $\mathbf{R} \rightarrow \mathbf{R}$ (addition, multiplication, and division in a field are all continuous for any absolute value on the field). When $|r|_p \leq 1$, r lies in \mathbf{Z}_p so r is a p -adic limit of nonnegative integers (use the truncations of the p -adic expansion of r): write $r = \lim_{i \rightarrow \infty} a_i$ where $a_i \in \mathbf{N}$. By p -adic continuity of polynomials,

$$\binom{r}{k} = \lim_{i \rightarrow \infty} \binom{a_i}{k}.$$

Each $\binom{a_i}{k}$ is an integer since binomial coefficients with nonnegative integers upstairs are integers ($\binom{a_i}{k} = 0$ if $0 \leq a_i < k$ and $\binom{a_i}{k} \in \mathbf{Z}^+$ if $a_i \geq k$ by combinatorics). Thus $\binom{a_i}{k} \in \mathbf{Z}_p$ for each i , and \mathbf{Z}_p is closed in \mathbf{Q}_p , so a p -adic limit of $\binom{a_i}{k}$ is in \mathbf{Z}_p . Hence $|\binom{r}{k}|_p \leq 1$. \square

Example 2. When the binomial coefficient $\binom{33/20}{7}$ is expanded out and simplified, the denominator can only have prime factors 2 and 5. Explicitly,

$$\binom{33/20}{7} = -\frac{352590381}{102400000000} = -\frac{352590381}{2^{18}5^8}.$$

The theorem we proved admits a converse.

Theorem 3. *Each prime p that divides the denominator of r also divides the denominator of every $\binom{r}{k}$ for $k \geq 1$.*

Obviously this doesn't hold at $k = 0$, since $\binom{r}{0} = 1$.

Proof. Let's reformulate the theorem in terms of p -adic absolute values: if $|r|_p > 1$ then $|\binom{r}{k}|_p > 1$ for all $k \geq 1$. The top of $\binom{r}{k}$ is $r(r-1)\cdots(r-(k-1))$, and for each positive integer j we have $|r-j|_p = |r|_p$ since $|r|_p > 1 \geq |j|_p$ so

$$\begin{aligned} \left| \binom{r}{k} \right|_p &= \left| \frac{r(r-1)(r-2)\cdots(r-(k-1))}{k!} \right|_p \\ &= \frac{|r|_p |r-1|_p |r-2|_p \cdots |r-(k-1)|_p}{|k!|_p} \\ &= \frac{|r|_p^k}{|k!|_p} \\ &= |r|_p^k p^{\text{ord}_p(k!)} \\ &\geq |r|_p^k \\ &> 1. \end{aligned}$$

\square

Example 4. Every Taylor coefficient of $\sqrt[3]{1+x}$ besides the constant term must have its reduced form denominator divisible by 2 and by 3; it can never be a power of just one of those primes.