THE SPLITTING FIELD OF $X^3 - 5$ OVER Q

KEITH CONRAD

In this note, we calculate all the basic invariants of the number field
$$K = \mathbf{Q}(\sqrt[3]{5}, \omega),$$
where $\omega = (-1 + \sqrt{-3})/2$ is a primitive cube root of unity.

Here is the notation for the fields and Galois groups to be used. Let

$$\begin{aligned} k &= \mathbf{Q}(\sqrt[3]{5}), \\ K &= \mathbf{Q}(\sqrt[3]{5}, \omega), \\ F &= \mathbf{Q}(\omega) = \mathbf{Q}(\sqrt{-3}), \\ G &= \mathrm{Gal}(K/\mathbf{Q}) \cong S_3, \\ N &= \mathrm{Gal}(K/F) \cong A_3, \\ H &= \mathrm{Gal}(K/k). \end{aligned}$$

First we work out the basic invariants for the fields F and k.

Theorem 1. *The field $F = \mathbf{Q}(\omega)$ has ring of integers $\mathbf{Z}[\omega]$, class number 1, discriminant -3, and unit group $\{\pm 1, \pm\omega, \pm\omega^2\}$. The ramified prime 3 factors as $3 = -(\sqrt{-3})^2$. For $p \neq 3$, the way p factors in $\mathbf{Z}[\omega] = \mathbf{Z}[X]/(X^2 + X + 1)$ is identical to the way $X^2 + X + 1$ factors mod p, so p splits if $p \equiv 1 \bmod 3$ and p stays prime if $p \equiv 2 \bmod 3$.*

We now turn to the field k. Its norm form is
$$\mathrm{N}_{k/\mathbf{Q}}(a + b\sqrt[3]{5} + c\sqrt[3]{25}) = a^3 + 5b^3 + 25c^3 - 15abc.$$

Since $\mathrm{disc}(\mathbf{Z}[\sqrt[3]{5}]) = -\mathrm{N}_{k/\mathbf{Q}}(3(\sqrt[3]{5})^2) = -3^3 5^2$, only 3 and 5 can ramify in k. Since $X^3 - 5$ is Eisenstein at 5 and

$$(X-1)^3 - 5 = X^3 - 3X^2 + 3X - 6 \tag{1}$$

is Eisenstein at 3, both 3 and 5 are totally ramified. Therefore by the same local field argument as in [2], $\mathcal{O}_k = \mathbf{Z}[\sqrt[3]{5}]$, so $\mathrm{disc}(\mathcal{O}_k) = -3^3 5^2$.

The factorization of 5 is $5\,\mathcal{O}_k = (\sqrt[3]{5})^3$. To factor 3, we use not (1) but a 3-Eisenstein polynomial whose constant term is ± 3:
$$(X-2)^3 + 5 = X^3 - 6X^2 + 12X - 3.$$

Let $\pi \stackrel{\text{def}}{=} 2 - \sqrt[3]{5}$ be a root of this. Then
$$\begin{aligned} \pi^3 &= 3 - 12\pi + 6\pi^2 \\ &= 3(1 - 4\pi + 2\pi^2) \\ &= 3(1 - 4\sqrt[3]{5} + 2\sqrt[3]{25}). \end{aligned}$$

Since $\mathrm{N}_{k/\mathbf{Q}}(\pi) = 3$, $v \stackrel{\text{def}}{=} 1 - 4\sqrt[3]{5} + 2\sqrt[3]{25}$ is a unit in $\mathcal{O}_k$ with norm 1 and $3 = \pi^3/v$.

Let
$$u \stackrel{\text{def}}{=} \frac{1}{v} = 41 + 24\sqrt[3]{5} + 14\sqrt[3]{25},$$

so $3 = \pi^3 u$. We will show later that u is the fundamental unit of k. (For comparison, in $\mathbf{Q}(\sqrt[3]{2}, \omega)$ we had $3 = \pi^3 v$ where $v < 1$ was the reciprocal of the fundamental unit.)

The minimal polynomial of v over $\mathbf{Q}$ is

$$m(T) = T^3 - \mathrm{Tr}_{k/\mathbf{Q}}(v)T^2 + \mathrm{Tr}_{k/\mathbf{Q}}(u)T - 1 = T^3 - 3T^2 + 123T - 1,$$

and the minimal polynomial for u over $\mathbf{Q}$ is

$$T^3 - 123T^2 + 3T - 1.$$

It is not a surprise that $m(T)$ has a large linear coefficient, since $m(0) = -\mathrm{N}_{k/\mathbf{Q}}(v) = -1$ but m has a zero $v \approx .008$ which is quite near 0, so $m'(0)$ ought to be large, in fact around $(m(v) - m(0))/v = 1/v \approx 1/.008 \approx 122.9$. Since the minimal polynomial for u has a root mod 2, $2 | \operatorname{disc}(\mathbf{Z}[u])$ so $\mathbf{Z}[u] \neq \mathcal{O}_k$. (In full, $\operatorname{disc}(\mathbf{Z}[u]) = -2^6 3^3 5^2 13^2$.)

To determine the class number of k, the Minkowski bound is

$$\frac{3!}{3^3}\left(\frac{4}{\pi}\right) 3 \cdot 5\sqrt{3} = \frac{40\sqrt{3}}{3\pi} < \frac{80}{3\pi} < \frac{27}{\pi} < 9.$$

So we must factor 2,3,5,7. We already saw 3 and 5 have principal prime factorizations. Since $X^3 - 5$ is irreducible mod 7, (7) stays prime in $\mathcal{O}_k$. Mod 2,

$$X^3 - 5 \equiv X^3 + 1 \equiv (X + 1)(X^2 + X + 1).$$

So $(2) = \mathfrak{p}\mathfrak{q}$, where $\mathrm{N}\,\mathfrak{p} = 2$ and $\mathrm{N}\,\mathfrak{q} = 4$.

Seeking principal generators for $\mathfrak{p}$ and $\mathfrak{q}$, we look for norms of elements divisible by 2. From $\mathrm{N}_{k/\mathbf{Q}}(1 + \sqrt[3]{5}) = 6$ we must have $(1 + \sqrt[3]{5}) = \mathfrak{p}(\pi)$, so we compute

$$\frac{1 + \sqrt[3]{5}}{\pi} = \frac{1 + \sqrt[3]{5}}{2 - \sqrt[3]{5}} = \frac{(1 + \sqrt[3]{5})(2 - \sqrt[3]{5}\omega)(2 - \sqrt[3]{5}\omega^2)}{3} = 3 + 2\sqrt[3]{5} + \sqrt[3]{25}.$$

This is a generator for $\mathfrak{p}$, from which we get a generator for $\mathfrak{q}$:

$$2 = (3 + 2\sqrt[3]{5} + \sqrt[3]{25})(-1 - \sqrt[3]{5} + \sqrt[3]{25}). \tag{2}$$

Thus k has class number 1.

Let $U > 1$ be the fundamental unit of k. As in [2, Lemma 2], $|\operatorname{disc}(\mathcal{O}_K)|/4 < U^3 + 7$, so

$$U^2 > \left(\frac{3^3 5^2}{4} - 7\right)^{2/3} \approx 29.6.$$

Alas, this is not greater than $u \approx 122.9$, so we can't conclude that $U^2 > u$, and hence that $U = u$. Yet $U = u$ is true. How can this be proven?

Theorem 2. *The fundamental unit of* $k = \mathbf{Q}(\sqrt[3]{5})$ *is* $u = 41 + 24\sqrt[3]{5} + 14\sqrt[3]{25}$.

Proof. We use a technique taken from the tome of Delone and Faddeev on cubic fields [3, pp. 88-92]. (For a table of cubic number field data, including fundamental units, see [3, pp. 141-146]. For a list of fundamental units of pure cubic fields $\mathbf{Q}(\sqrt[3]{m})$ with $m \leq 250$, see [4].)

We will show u is a fundamental unit by showing u is not an jth power of an algebraic integer in $\mathcal{O}_k$ for any $j > 1$.

Suppose $u = \rho^j$, where $\rho^3 + a\rho^2 + b\rho + c = 0$ for integers a, b, c. Since ρ must be some power of the fundamental unit, $c = -\mathrm{N}_{k/\mathbf{Q}}(\rho) = -1$.

The key idea we'll use is that symmetric functions in the $\mathbf{Q}$-conjugates of u are symmetric functions in the $\mathbf{Q}$-conjugates of ρ, and hence are integral polynomials in a and b (since $c = -1$ is known already). Studying such integral polynomials will impose conditions on the coefficients a, b.

We will denote the conjugates of u and ρ with prime notation, so

$$u + u' + u'' = 123, \quad uu' + uu'' + u'u'' = 3, \quad uu'u'' = 1,$$

$$\rho + \rho' + \rho'' = -a, \quad \rho\rho' + \rho\rho'' + \rho'\rho'' = b, \quad \rho\rho'\rho'' = -c = 1.$$

So if $u = \rho^j$ then $123 = \mathrm{Tr}_{k/\mathbf{Q}}(\rho^j) = F_j(a,b)$ and $3 = G_j(a,b)$ for some $F_j, G_j \in \mathbf{Z}[X,Y]$. When $j = 2$ and 3 we can work with F_j and G_j directly. But for larger j that becomes too cumbersome.

Let's suppose $u = \rho^2$. Then

$$123 = \rho^2 + (\rho')^2 + (\rho'')^2 = (\rho + \rho' + \rho'')^2 - 2(\rho\rho' + \rho\rho'' + \rho'\rho'') = a^2 - 2b$$

and

$$3 = (\rho\rho')^2 + (\rho\rho'')^2 + (\rho'\rho'')^2 = b^2 - 2ac = b^2 + 2a.$$

Solving for a in terms of b and feeding that into the first equation,

$$123 = \frac{(3-b^2)^2}{4} - 2b \Rightarrow b^4 - 6b^2 - 8b - 483 = 0.$$

Therefore $b | 483 = 3 \cdot 7 \cdot 23$, but none of the divisors is a root of the quartic polynomial. So u is not a square.

Now suppose $u = \rho^3$. In general,

$$x^3 + y^3 + z^3 = (x+y+z)^3 - 3(x+y+z)(xy+xz+yz) + 3(xyz).$$

Thus

$$123 = \rho^3 + (\rho')^3 + (\rho'')^3 = -a^3 + 3ab + 3$$

and

$$3 = (\rho\rho')^3 + (\rho\rho'')^3 + (\rho'\rho'')^3 = b^3 + 3ab + 3.$$

The second equation says $b = 0$ or $b^2 = -3a$. If $b = 0$, then $120 = -a^3$, which is impossible. So $b^2 = -3a$, hence

$$123 = b^6/27 - b^3 + 3 \Rightarrow b^6 - 27b^3 - 27 \cdot 120 = 0.$$

The roots of $T^2 - 27T - 27 \cdot 120$ are 72 and -45; neither is a cube. So u is not a cube.

Now suppose $u = \rho^p$ for an odd prime p. Then $u \pm 1$ is divisible by $\rho \pm 1$ in $\mathcal{O}_k$, so in $\mathbf{Z}$

$$\mathrm{N}_{k/\mathbf{Q}}(\rho+1) | \mathrm{N}_{k/\mathbf{Q}}(u+1) = 128, \quad \mathrm{N}_{k/\mathbf{Q}}(\rho-1) | \mathrm{N}_{k/\mathbf{Q}}(u-1) = 120. \tag{3}$$

Since $\rho > 1$, $\mathrm{N}_{k/\mathbf{Q}}(\rho \pm 1)$ is positive. From the cubic polynomial satisfied by ρ,

$$\mathrm{N}_{k/\mathbf{Q}}(\rho+1) = 1 - a + b - c = 2 - a + b, \quad \mathrm{N}_{k/\mathbf{Q}}(\rho-1) = -1 - a - b - c = -a - b. \tag{4}$$

By the symmetric function theorem,

$$123 = \rho^p + (\rho')^p + (\rho'')^p = (\rho + \rho' + \rho'')^p + pA \equiv -a^p \bmod p \equiv -a \bmod p \tag{5}$$

for some integer A, and similarly

$$3 \equiv (\rho\rho' + \rho\rho'' + \rho'\rho'')^p \equiv b^p \equiv b \bmod p. \tag{6}$$

By (4), the divisibility relations (3) concern not a and b but $2 - a + b$ and $-a - b$. For *odd* p, the congruences (5) and (6) are equivalent to

$$2 - a + b \equiv 128 \bmod p, \quad -a - b \equiv 120 \bmod p. \tag{7}$$

Coupled with the conditions

$$2 - a + b, \ -a - b \in \mathbf{Z}^+, 2 - a + b | 128, -a - b | 120, \tag{8}$$

we assemble a finite list of possibilities for $2 - a + b$ and for $-a - b$, along with the corresponding possibilities for p:

$2-a+b$	1	2	4	8	16	32	64	128
p	127	2,3, 7	2, 31	2,3,5	2, 7	2, 3	2	arb.

$-a-b$	1	2	3	4	5	6	8	10
p	7, 17	2, 59	3, 13	2, 29	5, 23	2, 3, 19	2, 7	2, 5, 11
$-a-b$	12	15	20	24	30	40	60	120
p	2, 3	3, 5, 7	2, 5	2,3	2, 3, 5	2, 5	2, 3, 5	arb.

Larger primes appear less often (only 2, 3, 5, and 7 appear more than once), so we consider primes from largest to smallest.

First, we handle the "arbitrary" case, when $2-a+b=128$ and $-a-b=120$. Then $a=-123, b=3$, so ρ is a root of T^3-123T^3+3T-1, i.e. $\rho=u$. This is useless.

If $p=127$ then $2-a+b=1$ and $-a-b=120$. There is no solution; $2-a+b$ and $-a-b$ have the same parity. Similarly, there is no solution when $p=23,17,13$.

If $p=59$ then $2-a+b=128$ and $-a-b=2$, so $a=-64, b=62$. We consider a root ρ of the polynomial $T^3-64T^2+62T-1$. If $\rho\in\mathcal{O}_k$ and $u=\rho^j$, then

$$u=\rho^j \Rightarrow \mathbf{Z}[u]\subset\mathbf{Z}[\rho]\subset\mathcal{O}_k \Rightarrow 3^35^2|\operatorname{disc}(\mathbf{Z}[\rho])|2^63^35^213^2.$$

Neither of these divisibility relations holds, since $T^3-64T^2+62T-1$ has discriminant equal to the prime 13814533.

We can similarly eliminate the possibility of other primes:

p	polynomial	discriminant
31	$T^3-61T^2-59T-1$	$2^4\cdot 59\cdot 71\cdot 191$
29	$T^3-65T^2+61T-1$	$2^5\cdot 43\cdot 233$
19	$T^3-66T^2+60T-1$	$3^3\cdot 508847$
11	$T^3-68T^2+58T-1$	$5^2\cdot 13\cdot 41809$

Now we need to handle the primes ≤ 7. The cases $p=2,3$ have already been treated, so 5 and 7 remain.

To eliminate 5 and 7 by constructing cubic polynomials from the tables above will require over 25 cases. Instead of pursuing this idea further, we show u is not a fifth or seventh power in $\mathcal{O}_k$ by showing it is not such a power in some residue field $\mathcal{O}_k/\mathfrak{p}\cong\mathbf{F}_p$.

To show u is not a fifth power in some $\mathbf{F}_p$, we want $5|p-1$, so let's try $p=11$. Since X^3-5 has a (single) root 3 mod 11, there is a prime ideal $\mathfrak{p}_{11}$ with norm 11. In $\mathcal{O}_k/\mathfrak{p}_{11}$,

$$u\equiv\rho^5\equiv\pm1\Rightarrow 11|\operatorname{N}_{k/\mathbf{Q}}(u\pm1).$$

Since $\operatorname{N}_{k/\mathbf{Q}}(u+1)=128$ and $\operatorname{N}_{k/\mathbf{Q}}(u-1)=120$, u is not a fifth power.

For seventh powers, we want $7|p-1$. Try $p=29$. Since X^3-5 has a (single) root -7 mod 29, there is a prime ideal $\mathfrak{p}_{29}$ with norm 29, and in its residue field

$$u\equiv\rho^7\Rightarrow u^2\equiv\rho^{14}\equiv\pm1.$$

We already know $u^2-1=(u-1)(u+1)$ has norm not divisible by 29. Since $\operatorname{N}_{k/\mathbf{Q}}(u^2+1)=2^3\cdot 1861$, u is not a seventh power. $\square$

Theorem 3. *The field $k=\mathbf{Q}(\sqrt[3]{5})$ has ring of integers $\mathcal{O}_k=\mathbf{Z}[\sqrt[3]{5}]$, class number 1, discriminant -3^35^2, and unit group $\pm u^{\mathbf{Z}}$ where $u=41+24\sqrt[3]{5}+14\sqrt[3]{25}$. Also $1/u=v=1-4\sqrt[3]{5}+2\sqrt[3]{25}$. The ramified primes 3 and 5 factor as $3=\pi^3u$ and $5=(\sqrt[3]{5})^3$, where $\pi=2-\sqrt[3]{5}$. The minimal polynomials of π and u are respectively*

$$T^3-6T^2+12T-3,\quad T^3-123T^2+3T-1.$$

We now turn to K. The only ramified primes are 3 and 5. Just as in [2], $(3)=(\eta)^6$ where $\eta=\sqrt{-3}/\pi$, so $\eta^2=-3/\pi^2=-\pi u$. (In [2], $\eta^2=-\pi v$.) To find the minimal polynomial of η over $\mathbf{Q}$, we work out the one for $\eta^2=-\pi u=-(12+7\sqrt[3]{5}+4\sqrt[3]{25})$:

$$\mathrm{N}_{k/\mathbf{Q}}(-\pi u)=-\mathrm{N}_{k/\mathbf{Q}}(\pi)=-3,\quad \mathrm{Tr}_{k/\mathbf{Q}}(-\pi u)=-36.$$

The linear coefficient in the minimal polynomial for $-\pi u$ is

$$3\,\mathrm{Tr}_{k/\mathbf{Q}}(1/\pi u)=3\,\mathrm{Tr}_{k/\mathbf{Q}}(\pi^2/3)=12,$$

so the minimal polynomial for $-\pi u$ us $T^3+36T^2+12T+1$, hence that for η is

$$T^6+36T^4+12T^2+3,$$

so $\mathrm{disc}(\mathbf{Z}[\eta])=-2^6 3^7 5^4 23^4$.

The discriminant of $K/\mathbf{Q}$ can be calculated locally using completions at η and at $\sqrt[3]{5}$ (which stays prime in K), but instead we can use [2, Corollary 1]:

$$\mathrm{disc}(K)=\mathrm{disc}(F)\,\mathrm{disc}(k)^2=-3^7 5^4.$$

The ring of integers of K is computed by the same technique as in [2], with a similar result:

$$\mathcal{O}_K=\mathcal{O}_k\oplus\mathcal{O}_k\,\theta,$$

where $\theta=(\omega-1)/\pi$, so $\eta=-\omega\theta$. Since

$$\theta\overline{\theta}=\frac{3}{\pi^2}=\pi u=12+7\sqrt[3]{5}+4\sqrt[3]{25},\quad \theta+\overline{\theta}=-\frac{3}{\pi}=-\pi^2 u=-(4+2\sqrt[3]{5}+\sqrt[3]{25}),$$

the minimal polynomial of θ over k is

$$f(T)=T^2+\pi^2 uT+\pi u=T^2+(4+2\sqrt[3]{5}+\sqrt[3]{25})T+(12+7\sqrt[3]{5}+4\sqrt[3]{25}),$$

so the minimal polynomial of θ over $\mathbf{Q}$ is

$$f\sigma(f)\sigma^2(f)=T^6+12T^5+54T^4+72T^3+48T^2+18T+3,$$

where $\sigma\in N=\mathrm{Gal}(K/F)$ is an element of order 3. This polynomial has discriminant $-2^8 3^7 5^4$, so $\mathcal{O}_k\neq\mathbf{Z}[\theta]$. Also $\mathcal{O}_K\neq\mathbf{Z}[\eta]$.

Now we turn to class number computations. The Minkowski bound for K is

$$\frac{6!}{6^6}\left(\frac{4}{\pi}\right)^3 5^2 3^3\sqrt{3}=\frac{2^4 5^3\sqrt{3}}{3\pi^3}\approx 37.2.$$

The factorization statements in [2] for $\mathbf{Q}(\sqrt[3]{2},\omega)$ apply similarly to $K=\mathbf{Q}(\sqrt[3]{5},\omega)$, so the only possible rational primes which don't factor principally in K are those $p\equiv 1 \bmod 3$ where 5 is a cube mod p, and such primes split completely in K. There is one prime ≤ 37 with these properties, $p=13$, so $\mathrm{Cl}(K)$ is generated by the prime ideal factors of 13. Since $\mathrm{N}_{K/\mathbf{Q}}(\theta-1)=g(1)=208=2^4\cdot 13$, there is a principal prime ideal factor of 13, so $h(K)=1$.

(For the interested reader, we compute an explicit generator of a prime ideal over 13 by factoring $(\theta-1)$.

The factorization of 2 is $2\,\mathcal{O}_K=\mathfrak{p}\sigma\mathfrak{p}\sigma^2\mathfrak{p}$, where $\mathfrak{p}=(3+2\sqrt[3]{5}+\sqrt[3]{25})$, and $f_2(K/\mathbf{Q})=2$. Which of $\mathfrak{p}$ and its conjugates divides $(\theta-1)$? All three ideals have quotient $\mathbf{F}_4$, so the cube roots of unity are all distinct in the corresponding residue fields.

In $\mathcal{O}_K/\mathfrak{p}$, $1+\sqrt[3]{25}\equiv 0\Rightarrow\sqrt[3]{5}\equiv 1\Rightarrow\theta\equiv\omega-1\equiv\omega^2\not\equiv 1$.

In $\mathcal{O}_K/\sigma\mathfrak{p}$, $1+\sqrt[3]{25}\omega^2\equiv 0\Rightarrow\sqrt[3]{5}\equiv\omega^2\Rightarrow\theta\equiv(\omega-1)/\omega^2\equiv 1$.

In $\mathcal{O}_K/\sigma^2\mathfrak{p}$, $1+\sqrt[3]{25}\omega\equiv 0\Rightarrow\sqrt[3]{5}\equiv\omega\Rightarrow\theta\equiv(\omega-1)/(-\omega)\equiv\omega\not\equiv 1$.

Therefore $(\theta-1)=(\sigma\mathfrak{p})^2\mathfrak{P}_{13}$, where $\mathfrak{P}_{13}|(13)$, so $\mathfrak{P}_{13}$ is a principal ideal with

$$\beta\stackrel{\mathrm{def}}{=}\frac{\theta-1}{(3+2\sqrt[3]{5}\omega+\sqrt[3]{25}\omega^2)^2}=-(9+10\sqrt[3]{5}+5\sqrt[3]{25}+(1+7\sqrt[3]{5}-\sqrt[3]{25})\theta)$$

as a generator.)

Now we turn to the unit group of $\mathcal{O}_K$. Since the ideal (η) is fixed by the Galois group of $K/\mathbf{Q}$, let's consider the unit

$$\delta \stackrel{\text{def}}{=} \frac{\sigma(\eta)}{\eta} = \frac{\pi}{\sigma(\pi)} \in \mathcal{O}_K^\times,$$

where $\sigma \in \mathrm{Gal}(K/F)$ sends $\sqrt[3]{5}$ to $\sqrt[3]{5}\omega$. We have

$$\pi = 2 - \sqrt[3]{5}, \quad \sigma(\pi) = 2 - \sqrt[3]{5}\omega, \quad \sigma^2(\pi) = 2 - \sqrt[3]{5}\omega^2 = \overline{\sigma}(\pi).$$

Therefore

$$\overline{\delta} = \frac{\pi}{\overline{\sigma}(\pi)} = \frac{\pi}{\sigma^2\pi},$$

so

$$|\delta|^2 = \delta\overline{\delta} = \frac{\pi^2}{\sigma(\pi)\sigma^2(\pi)} = \frac{\pi^3}{3} = v,$$

so

$$v = \mathrm{N}_{K/k}(\delta), \quad u = \mathrm{N}_{K/k}(1/\delta).$$

The log map on $\mathcal{O}_K^\times$ is given by

$$L(x) = (2\log|x|, 2\log|\sigma(x)|, 2\log|\sigma^2(x)|).$$

We compute this for $x = u, \delta, \sigma(\delta)$, keeping only the first two coordinates.

Since $\mathrm{N}_{k/\mathbf{Q}}(u) = u\sigma(u)\overline{\sigma}(u) = u|\sigma(u)|^2$, $2\log|\sigma(u)| = 2\log|\sigma^2(u)| = -\log u$.

Since

$$\sigma(\delta) = \frac{\sigma(\pi)}{\sigma^2(\pi)}, \quad \sigma^2(\delta) = \frac{\sigma^2(\pi)}{\pi},$$

we get

$$2\log|\sigma(\delta)| = 0, \quad 2\log|\sigma^2(\delta)| = -2\log|\delta| = -\log v = \log u,$$

so

$$\begin{aligned} L(u) &= (2\log u, -\log u), \\ L(\sigma u) &= (-\log u, -\log u), \\ L(\delta) &= (-\log u, 0), \\ L(\overline{\delta}) &= (-\log u, \log u), \\ L(\sigma(\delta)) &= (0, \log u). \end{aligned}$$

In particular, notice that $L(\sigma(\delta)) = L(\overline{\delta}) - L(\delta)$, which means $\sigma(\delta) = \zeta\overline{\delta}/\delta$, where ζ is a root of unity in K; in fact $\sigma(\delta) = \overline{\delta}/\delta$. The regulator computations are:

unit pair	regulator
u, δ	$(\log u)^2$
δ, $\overline{\delta}$	$(\log u)^2$
u, $\sigma(u)$	$3(\log u)^2$

By [2, Cor. 1], $h(K)R(K) = h(F)R(F)(h(k)R(k))^2 = (\log u)^2$, so

$$[\mathcal{O}_K^\times/\mu_K : \langle\delta, \overline{\delta}\rangle] = \frac{\mathrm{Reg}(\delta, \overline{\delta})}{R(K)} = \frac{(\log u)^2}{R(K)} = h(K).$$

We already checked $h(K) = 1$, so $\{\delta, \overline{\delta}\}$ is a pair of fundamental units for K.

To match the notation for fundmental units in [2], let

$$\varepsilon \stackrel{\text{def}}{=} \omega^2\delta = \omega^2\frac{\pi}{\sigma(\pi)} = -7 + 4\sqrt[3]{5} + (7\sqrt[3]{5} - 12)\theta = 1 - 4\pi + (2 - 7\pi)\theta.$$

We know $\{\varepsilon, \overline{\varepsilon}\}$ is a pair of fundamental units. Might $\mathcal{O}_K = \mathbf{Z}[\varepsilon]$? Let's find the polynomial for ε over k, and then descend to $\mathbf{Q}$.

We compute

$$\mathrm{Tr}_{K/k}(\varepsilon)=\omega^2\frac{\pi}{\sigma(\pi)}+\omega\frac{\pi}{\sigma^2(\pi)}=\frac{\pi^2}{3}(\omega^2\sigma^2(\pi)+\omega\sigma(\pi))=\frac{\pi^2}{3}(-\pi)=-v.$$

So ε and $\overline{\varepsilon}$ are both roots of $f(T)=T^2+vT+v$. (This is analogous to the role of the polynomial T^2+uT+u in [2].) So the minimal polynomial of ε over $\mathbf{Q}$ is

$$f\sigma(f)\sigma^2(f)=T^6+3T^5+126T^4+247T^3+126T^2+3T+1.$$

Alas, the discriminant of this is $-2^{12}3^75^413^6$, so $\mathcal{O}_K\neq\mathbf{Z}[\varepsilon]$. (As an aside, the polynomial has symmetric coefficients, so ε^{-1} is a root, and in fact $\varepsilon^{-1}=\overline{\sigma}^2(\varepsilon)$.)

Theorem 4. *The field $K=\mathbf{Q}(\sqrt[3]{5},\omega)$ has class number 1, discriminant -3^75^4, and regulator $(\log(41+24\sqrt[3]{5}+14\sqrt[3]{25}))^2$. The ramified primes 3 and 5 factor as*

$$(3)=(\eta)^6,\quad (5)=(\sqrt[3]{5})^3,$$

where $\eta=\sqrt{-3}/\pi$, $\pi=2-\sqrt[3]{5}$.

The ring of integers of K is $\mathcal{O}_k\oplus\mathcal{O}_k\,\theta$, where $\theta=(\omega-1)/\pi$. The unit group of $\mathcal{O}_K$ has six roots of unity, rank 2, and basis $\{\varepsilon,\overline{\varepsilon}\}$, where

$$\varepsilon=\omega^2\pi/\sigma(\pi)$$

has minimal polynomial

$$g(T)=T^6+3T^5+126T^4+247T^3+126T^2+3T+1.$$

There is no power basis for $\mathcal{O}_K$. For a more general result, see [1].

We now return to the computation of $\mathrm{Cl}(K)$. We noted that $\mathrm{Cl}(K)$ is generated by the prime ideal factors of 13, and then showed those factors are principal, using the special element θ. Here is an alternative computation of $h(K)=1$ which does not depend on knowing about θ.

Let's assume $h(K)\neq 1$, i.e. none of the prime ideals over 13 in K is principal. Then the Galois group of $K/\mathbf{Q}$ acts transitively on the nonidentity classes of $\mathrm{Cl}(K)$, and we show by this action that $h(K)=3$ if $h(K)>1$.

Let $\mathfrak{P}$ be one prime ideal in K lying over 13. Let τ denote complex conjugation, so $\tau\sigma=\sigma^2\tau$. Since k has class number 1, $\mathfrak{P}\tau(\mathfrak{P})\sim 1$. Therefore

$$\sigma(\mathfrak{P})\tau(\sigma(\mathfrak{P}))\sim 1\Rightarrow\sigma(\mathfrak{P})\sigma^2(\tau\mathfrak{P})\sim 1\Rightarrow\mathfrak{P}\sigma(\tau\mathfrak{P})\sim 1.$$

Therefore $\tau\mathfrak{P}\sim\sigma(\tau\mathfrak{P})$, so $\tau\sigma\tau(\mathfrak{P})\sim 1$. So $[\mathfrak{P}]\in\mathrm{Cl}(K)$ is fixed by $\tau\sigma\tau=\sigma^2$, so its stabilizer subgroup is either $\{1,\sigma,\sigma^2\}$ or G. Thus the number of nonidentity elements in $\mathrm{Cl}(K)$ is 1 or 2, so $h(K)=2$ or 3. Since

$$\mathfrak{P}\sigma(\mathfrak{P})\sigma^2(\mathfrak{P})=\mathrm{N}_{K/F}(\mathfrak{P})=(1\pm 2\sqrt{-3})\sim 1,$$

$[\mathfrak{P}]^3=1$, hence $3|h(K)$. So if $h(K)>1$ then $\mathrm{Cl}(K)=\{1,[\mathfrak{P}],[\tau\mathfrak{P}]\}$ is cyclic of size 3.

We saw earlier that $[\mathcal{O}_K^\times/\mu_K:\langle\delta,\overline{\delta}\rangle]=h(K)$. Assume $h(K)=3$. We shall apply the results in [2, Thm. 4] about index 3 sublattices of $\mathbf{Z}^2$. In particular, neither $L(\delta)$ nor $L(\overline{\delta})$ is in $3L$, so if the index is 3 then there is a basis $\{\delta,\xi\}$ of $\mathcal{O}_K^\times/\mu_K$, where

$$\delta\overline{\delta}=\zeta\xi^3\quad\text{or}\quad\delta/\overline{\delta}=\zeta\xi^3$$

for some root of unity ζ. Applying $\mathrm{N}_{K/k}$ to the first possibility yields $v^2=(\mathrm{N}_{K/k}(\xi))^3$ in $\mathcal{O}_k^\times$, which is absurd since v is a generator of $\mathcal{O}_k^\times$. Applying the log map to the second possibility yields

$$L(\delta)-L(\overline{\delta})=-L(\sigma(\delta))\in 3L,$$

so by Galois action we have $L(\delta), L(\overline{\delta}) \in 3L$, a contradiction of $[L(\mathcal{O}_K^\times) : L(\delta)\mathbf{Z}+L(\overline{\delta})\mathbf{Z}] = 3$. Therefore $h(K) = 1$.

Here's another point of view on the link between $h(K) = 1$ and principal factorization of $13\,\mathcal{O}_K$. Since $\mathrm{N}_{k/\mathbf{Q}}(2+\sqrt[3]{5}) = 13$,

$$13 = (2+\sqrt[3]{5})(2+\sqrt[3]{5}\omega)(2+\sqrt[3]{5}\omega^2) = (2+\sqrt[3]{5})(4-2\sqrt[3]{5}+\sqrt[3]{25}). \tag{9}$$

We want to factor the second term on the right in $\mathcal{O}_k$. Since $\mathrm{N}_{k/\mathbf{Q}}(1+\sqrt[3]{25}) = 26$ and $h(k) = 1$, by (2) we must have a numerical factorization

$$1+\sqrt[3]{25} = (3+2\sqrt[3]{5}+\sqrt[3]{25})(a+b\sqrt[3]{5}+c\sqrt[3]{25})$$

for some $a, b, c \in \mathbf{Z}$. Multiplying the two terms on the right we get a solution $a = -3, b = -2, c = 0$, i.e. $\mathrm{N}_{k/\mathbf{Q}}(-3+2\sqrt[3]{5}) = 13$. Guided by (9), we divide $-3+2\sqrt[3]{5}$ into $4-2\sqrt[3]{5}+\sqrt[3]{25}$ to get the principal (in fact, numerical) factorization of 13 in $\mathbf{Z}[\sqrt[3]{5}]$:

$$13 = (2+\sqrt[3]{5})(-3+2\sqrt[3]{5})(2+2\sqrt[3]{5}+\sqrt[3]{25}). \tag{10}$$

So 13 has principal prime factors in $\mathcal{O}_K$ if and only if the ideal $(2+\sqrt[3]{5})$ of k is the norm of a principal ideal in K, i.e. there is some $\alpha \in \mathcal{O}_K$ such that

$$\mathrm{N}_{K/k}(\alpha) = \pm(2+\sqrt[3]{5})u^m$$

for some $m \in \mathbf{Z}$. The norm must be positive, so the plus sign must hold. Since $u = \mathrm{N}_{K/k}(1/\delta)$, $h(K) = 1$ if and only if $2+\sqrt[3]{5}$ is a norm from K.

To explicitly exhibit $2+\sqrt[3]{5}$ as a norm from K, we consider the generator β of one of the prime factors of $13\,\mathcal{O}_K$. Does $\mathrm{N}_{K/k}(\beta) = 2+\sqrt[3]{5}$? No, since $\mathrm{N}_{K/k}(\beta) = 342+200\sqrt[3]{5}+117\sqrt[3]{25}$, which is much larger than $2+\sqrt[3]{5}$. By (10), $\mathrm{N}_{K/k}(\beta)$ must equal $(2+\sqrt[3]{5})u^m$, $(-3+2\sqrt[3]{5})u^m$, or $(2+2\sqrt[3]{5}+\sqrt[3]{25})u^m$ for some integer m. Taking logarithms to check in each case whether the unknown m is an integer, we find that

$$\mathrm{N}_{K/k}(\beta) = (2+2\sqrt[3]{5}+\sqrt[3]{25})u.$$

The prime ideals in $\mathcal{O}_K$ lying over $(2+\sqrt[3]{5})$ and $(2+2\sqrt[3]{5}+\sqrt[3]{25})$ are conjugate by σ or σ^2, so let's consider $\mathrm{N}_{K/k}(\sigma\beta)$. Using PARI, $\sigma(\theta) = -4+\sqrt[3]{25}-(6-2\sqrt[3]{25})\theta$, from which we compute

$$\mathrm{N}_{K/k}(\sigma\beta) = \sigma(\beta)\overline{\sigma}(\beta) = (2+\sqrt[3]{5})u^{-2}.$$

Thus

$$2+\sqrt[3]{5} = \mathrm{N}_{K/k}(\sigma\beta)u^2 = \mathrm{N}_{K/k}((\sigma\beta)/\delta^2). \tag{11}$$

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