Using tensor products, we will give a slick proof that any two splitting fields of a polynomial are (non-canonically) isomorphic over the base field.

**Theorem 1.** Let $K$ be a field and $f(X) \in K[X]$ be nonconstant. Any two splitting fields of $f(X)$ over $K$ are $K$-isomorphic.

**Proof.** Let $n = \deg f \geq 1$ and let $L_1$ and $L_2$ be splitting fields of $f(X)$ over $K$, so

$$L_1 = K(\alpha_1, \ldots, \alpha_n), \quad L_2 = K(\beta_1, \ldots, \beta_n),$$

where the $\alpha_i$'s and $\beta_j$'s are full sets of roots of $f(X)$. (Some $\alpha_i$'s and some $\beta_j$'s may be repeated since $f(X)$ might not be separable.) We want to show there is a field isomorphism $L_1 \to L_2$ which fixes the elements of $K$.

Since $L_1$ and $L_2$ are not zero, the ring $L_1 \otimes_K L_2$ is not zero because the tensor product of nonzero vector spaces is not zero. Since $L_1/K$ and $L_2/K$ are algebraic, we can write $L_1 = K[\alpha_1, \ldots, \alpha_n]$ and $L_2 = K[\beta_1, \ldots, \beta_n]$. Thus $L_1 \otimes_K L_2$ is generated as a $K$-algebra by the $2n$ elementary tensors $\{\alpha_i \otimes 1, 1 \otimes \beta_j\}$. Pick a maximal ideal $m$ in $L_1 \otimes_K L_2$ and consider the composite map

$$L_1 \to L_1 \otimes_K L_2 \to (L_1 \otimes_K L_2)/m,$$

where the first map is $x \mapsto x \otimes 1$ and the second map is the natural reduction. Both are $K$-algebra homomorphisms, so the composite is as well. Since $L_1$ is a field, the composite map is injective, so we can regard $(L_1 \otimes_K L_2)/m$ as a field extension of $L_1$. The $\alpha_i$'s are a full set of roots of $f(X)$ in $L_1$, so the only roots of $f(X)$ in $(L_1 \otimes_K L_2)/m$ are the $\alpha_i \otimes 1$ mod $m$. Each $1 \otimes \beta_j$ mod $m$ is a root of $f(X)$, so $1 \otimes \beta_j \equiv \alpha_i \otimes 1$ mod $m$ for some $i$. Therefore $(L_1 \otimes_K L_2)/m$ is generated as a $K$-algebra by all $\alpha_i \otimes 1$ mod $m$, which proves the above map $L_1 \to (L_1 \otimes_K L_2)/m$ is surjective, and hence is a $K$-algebra isomorphism.

We get a $K$-algebra isomorphism $L_2 \to (L_1 \otimes_K L_2)/m$ in a similar way. Composing $L_1 \to (L_1 \otimes_K L_2)/m$ with the inverse of $L_2 \to (L_1 \otimes_K L_2)/m$ gives us a $K$-algebra isomorphism from $L_1$ to $L_2$. $\square$

**Remark 2.** Each $\alpha_i \otimes 1$ and $1 \otimes \beta_j$ in $L_1 \otimes_K L_2$ is a solution to $f(t) = 0$. This typically gives us $2n$ solutions to $f = 0$ in $L_1 \otimes_K L_2$ when $f(X)$ is separable, so we should anticipate a collapsing of these roots into each other when we reduce $L_1 \otimes_K L_2$ modulo a maximal ideal and get a field, where $f(X)$ always has at most $n$ roots.

It might at first seem curious that the construction of a $K$-algebra isomorphism $L_1 \to L_2$ succeeded using any maximal ideal in $L_1 \otimes_K L_2$. In fact, different maximal ideals provide us with all the different isomorphisms. Let’s look at an example before proving the general result.

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1This isn’t always true: if $\alpha_i \in K$ then $\alpha_i$ is some $\beta_j$ and $\alpha_i \otimes 1 = 1 \otimes \alpha_i$. 
Example 3. Two splitting fields for $X^2 - 2$ over $\mathbb{Q}$ are $L_1 = \mathbb{Q}[T]/(T^2 - 2)$ and $L_2 = \mathbb{Q}(\sqrt{2})$ (a subfield of $\mathbb{R}$). There are two $\mathbb{Q}$-isomorphisms $L_1 \rightarrow L_2$, determined by the identification of $T$ in $L_1$ with $\pm \sqrt{2}$ in $L_2$. The tensor product of $L_1$ and $L_2$ over $\mathbb{Q}$ is

$$L_1 \otimes_{\mathbb{Q}} L_2 = \mathbb{Q}[T]/(T^2 - 2) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{2}) \cong \mathbb{Q}(\sqrt{2})[T]/(T^2 - 2) = \mathbb{Q}(\sqrt{2})[T]/(T - \sqrt{2})(T + \sqrt{2}).$$

Using the Chinese remainder theorem,

$$\mathbb{Q}(\sqrt{2})[T]/(T - \sqrt{2})(T + \sqrt{2}) \cong \mathbb{Q}(\sqrt{2})[T]/(T - \sqrt{2}) \times \mathbb{Q}(\sqrt{2})[T]/(T + \sqrt{2}) \cong \mathbb{Q}(\sqrt{2}) \times \mathbb{Q}(\sqrt{2}),$$

where $T$ on the left corresponds to $(\sqrt{2}, -\sqrt{2})$ on the right. The ring $\mathbb{Q}(\sqrt{2}) \times \mathbb{Q}(\sqrt{2})$ has two maximal ideals, $\{0\} \times \mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{2}) \times \{0\}$. The quotient by each of these maximal ideals is isomorphic to $\mathbb{Q}(\sqrt{2})$, with one sending $T$ to $\sqrt{2}$ and the other sending $T$ to $-\sqrt{2}$.

Theorem 4. With notation as in the proof of Theorem 1, the set of maximal ideals in $L_1 \otimes_K L_2$ is in bijection with the set of $K$-algebra isomorphisms $L_1 \rightarrow L_2$.

Proof. We want to describe a bijection between the sets

$$\{\text{K-algebra isomorphisms } L_1 \rightarrow L_2\} \leftrightarrow \{\text{Maximal ideals in } L_1 \otimes_K L_2\}.$$

From $K$-algebra isomorphism to maximal ideal: Let $L_1 \xrightarrow{\varphi} L_2$ be a $K$-algebra isomorphism. To construct from $\varphi$ a maximal ideal in $L_1 \otimes_K L_2$, we will construct a homomorphism from $L_1 \otimes_K L_2$ onto the field $L_2$ and then take its kernel. The function $L_1 \times L_2 \rightarrow L_2$ where $(x, y) \mapsto \varphi(x)y$ is $K$-bilinear, so there is a $K$-linear map

$$L_1 \otimes_K L_2 \xrightarrow{f_{\varphi}} L_2,$$

where $f_{\varphi}(x \otimes y) = \varphi(x)y$. This is onto since $f_{\varphi}(1 \otimes y) = y$. A computation shows $f_{\varphi}$ is multiplicative on products of elementary tensors, so $f_{\varphi}$ is a $K$-algebra homomorphism. Since $f_{\varphi}$ is surjective and $L_2$ is a field, the kernel of $f_{\varphi}$ is a maximal ideal. Set $M_{\varphi} = \ker f_{\varphi}$.

From maximal ideal to $K$-algebra isomorphism: Let $m$ be a maximal ideal in $L_1 \otimes_K L_2$. We will construct from $m$ a $K$-algebra isomorphism $L_1 \rightarrow L_2$. By the proof of Theorem 1, the natural composite maps

$$L_1 \rightarrow L_1 \otimes_K L_2 \rightarrow (L_1 \otimes_K L_2)/m \text{ and } L_2 \rightarrow L_1 \otimes_K L_2 \rightarrow (L_1 \otimes_K L_2)/m$$

are $K$-algebra isomorphisms. Call the first composite map $\psi_{1,m}$ and call the second one $\psi_{m}$. Set $\psi_{m} = \psi_{2,m}^{-1} \circ \psi_{1,m}$, so $\psi_{m}$ is a $K$-algebra isomorphism from $L_1$ to $L_2$.

We will now show $\varphi \sim M_{\varphi}$ and $m \sim \psi_{m}$ are inverses of each other: $\psi_{M_{\varphi}} = \varphi$ and $M_{\psi_{m}} = m$.

Starting with $\varphi$, that $\psi_{M_{\varphi}} = \varphi$ means $\psi_{1,M_{\varphi}} = \psi_{2,M_{\varphi}} \circ \varphi$ as maps $L_1 \rightarrow (L_1 \otimes_K L_2)/M_{\varphi}$. For

$$\psi_{1,M_{\varphi}} : L_1 \rightarrow L_1 \otimes_K L_2 \rightarrow (L_1 \otimes_K L_2)/M_{\varphi},$$

the effect is $x \mapsto x \otimes 1 \mapsto x \otimes 1 \bmod M_{\varphi}$. For

$$\psi_{2,M_{\varphi}} \circ \varphi : L_1 \rightarrow L_2 \rightarrow L_1 \otimes_K L_2 \rightarrow (L_1 \otimes_K L_2)/M_{\varphi}$$

the effect is $x \mapsto \varphi(x) \mapsto 1 \otimes \varphi(x) \mapsto 1 \otimes \varphi(x) \bmod M_{\varphi}$. Therefore we need to show $x \otimes 1 \equiv 1 \otimes \varphi(x) \bmod M_{\varphi}$. Recall that $M_{\varphi} = \ker f_{\varphi}$, so this congruence amounts to saying $f_{\varphi}(x \otimes 1) = f_{\varphi}(1 \otimes \varphi(x))$. From the definition of $f_{\varphi}$ we have $f_{\varphi}(x \otimes 1) = \varphi(x) \cdot 1 = \varphi(x)$ and $f_{\varphi}(1 \otimes \varphi(x)) = \varphi(1)\varphi(x) = \varphi(x)$.
Starting with $m$, that $M_{\psi_m} = m$ means $\ker f_{\psi_m} = m$. We will show the diagram

\[
\begin{array}{ccc}
L_1 \otimes_K L_2 & \xrightarrow{\text{redn.}} & (L_1 \otimes_K L_2)/m \\
\downarrow f_{\psi_m} & & \\
L_2 & \xrightarrow{\psi_{2,m}} & (L_1 \otimes_K L_2)/m \\
\end{array}
\]

commutes. Then since $\psi_{2,m}$ is an isomorphism, the kernels of the two maps out of $L_1 \otimes_K L_2$ would be equal, so $\ker f_{\psi_m} = m$.

To verify commutativity of the diagram, it suffices (by additivity of all the maps) to focus on elementary tensors $x \otimes y$ in $L_1 \otimes_K L_2$, where we want to check

\[
\psi_{2,m}(f_{\psi_m}(x \otimes y)) \equiv x \otimes y \mod m.
\]

The left side is

\[
\psi_{2,m}(f_{\psi_m}(x \otimes y)) = \psi_{2,m}(\psi_m(x)y) = \psi_{2,m}(\psi_m(x))\psi_{2,m}(y) = (\psi_{2,m} \circ \psi_m)(x)\psi_{2,m}(y) = \psi_{1,m}(x)\psi_{2,m}(y) = (x \otimes 1) \mod m \cdot (1 \otimes y) \mod m = x \otimes y \mod m.
\]