# SIMPLE RADICAL EXTENSIONS

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## 1. Introduction

A field extension $L/K$ is called *simple radical* if $L = K(\alpha)$ where $\alpha^n = a$ for some $n \geq 1$ and $a \in K^\times$. Examples of simple radical extensions of $\mathbf{Q}$ are $\mathbf{Q}(\sqrt{2})$, $\mathbf{Q}(\sqrt[3]{2})$, and more generally $\mathbf{Q}(\sqrt[n]{2})$. A root of $T^n - a$ will be denoted $\sqrt[n]{a}$, so a simple radical extension of $K$ looks like $K(\sqrt[n]{a})$, but the notation $\sqrt[n]{a}$ in general fields is ambiguous: different $n$th roots of $a$ can generate different extensions of $K$, and they could even be nonisomorphic (*e.g.*, have different degrees over $K$) if $T^n - a$ is reducible in $K[T]$.

**Example 1.1.** In $\mathbf{C}$ the three roots of $T^3 - 8$ are 2, $2\omega$, and $2\omega^2$, where $\omega$ is a nontrivial cube root of unity; note $\omega^2 = 1/\omega$ and $\omega$ is a root of $(T^3-1)/(T-1) = T^2 + T + 1$. While $\mathbf{Q}(2) = \mathbf{Q}$, the extension $\mathbf{Q}(2\omega) = \mathbf{Q}(\omega) = \mathbf{Q}(2/\omega)$ has degree 2 over $\mathbf{Q}$, so when the notation $\sqrt[3]{8}$ denotes any of the roots of $T^3 - 8$ over $\mathbf{Q}$ then the field $\mathbf{Q}(\sqrt[3]{8})$ has two different meanings and $\mathbf{R}(\sqrt[3]{8})$ is $\mathbf{R}$ if $\sqrt[3]{8} = 2$ and it is $\mathbf{C}$ if $\sqrt[3]{8}$ is $2\omega$ or $2\omega^2$.

**Example 1.2.** In the field $\mathbf{Q}(\sqrt{5})$ the number $2 + \sqrt{5}$ is a cube: $2 + \sqrt{5} = (\frac{1+\sqrt{5}}{2})^3$. The polynomial $T^3 - (2+\sqrt{5})$ factors over $\mathbf{Q}(\sqrt{5})$ as

$$T^3 - (2+\sqrt{5}) = \left(T - \frac{1+\sqrt{5}}{2}\right)\left(T^2 + \frac{1+\sqrt{5}}{2}T + \frac{3+\sqrt{5}}{2}\right)$$

and the second factor is irreducible over $\mathbf{Q}(\sqrt{5})$ since it is irreducible over the larger field $\mathbf{R}$ (it is a quadratic with negative discriminant $-3(3+\sqrt{5})/2$). If $\sqrt[3]{2+\sqrt{5}}$ means $(1+\sqrt{5})/2$ then $\mathbf{Q}(\sqrt[3]{2+\sqrt{5}}) = \mathbf{Q}((1+\sqrt{5})/2) = \mathbf{Q}(\sqrt{5})$, and if $\sqrt[3]{2+\sqrt{5}}$ is a root of the quadratic factor of $T^3 - (2+\sqrt{5})$ above then $\mathbf{Q}(\sqrt[3]{2+\sqrt{5}})$ is a quadratic extension of $\mathbf{Q}(\sqrt{5})$.

We will focus here on the degree $[K(\sqrt[n]{a}) : K]$ and irreducibility relations for $T^n - a$ among different values of $n$, and intermediate fields between $K$ and $K(\sqrt[n]{a})$.

## 2. Basic properties of $T^n - a$ and $\sqrt[n]{a}$

**Theorem 2.1.** *The degree $[K(\sqrt[n]{a}) : K]$ is at most $n$, and it equals $n$ if and only if $T^n - a$ is irreducible over $K$, in which case the field $K(\sqrt[n]{a})$ up to isomorphism is independent of the choice of $\sqrt[n]{a}$.*

*Proof.* Since $\sqrt[n]{a}$ is a root of $T^n - a$, which is in $K[T]$, the minimal polynomial of $\sqrt[n]{a}$ over $K$ is at most $n$, and thus $[K(\sqrt[n]{a}) : K] \leq n$.

If $[K(\sqrt[n]{a}) : K] = n$ then the minimal polynomial of $\sqrt[n]{a}$ over $K$ has degree $n$, so it must be $T^n - a$ since that polynomial has degree $n$ in $K[T]$ with $\sqrt[n]{a}$ as a root. As a minimal polynomial in $K[T]$ for some number, $T^n - a$ is irreducible over $K$.

Conversely, assume $T^n - a$ is irreducible over $K$. Then $\sqrt[n]{a}$ has minimal polynomial $T^n - a$ over $K$ (the minimal polynomial of a number over $K$ is the unique monic irreducible polynomial in $K[T]$ with that number as a root), so $[K(\sqrt[n]{a}) : K] = \deg(T^n - a) = n$.

When $T^n - a$ is irreducible over $K$, the field $K(\sqrt[n]{a})$ is isomorphic to $K[T]/(T^n - a)$ using evaluation at $\sqrt[n]{a}$ and thus, up to isomorphism (not up to equality!), the field $K(\sqrt[n]{a})$ is independent of the choice of $\sqrt[n]{a}$. □

**Example 2.2.** The polynomial $T^3 - 2$ is irreducible over $\mathbf{Q}$ and the three fields $\mathbf{Q}(\sqrt[3]{2})$, $\mathbf{Q}(\sqrt[3]{2}\omega)$, and $\mathbf{Q}(\sqrt[3]{2}\omega^2)$ are isomorphic to each other, where $\sqrt[3]{2}$ is the real cube root of 2 (or any cube root of 2 in characteristic 0) and $\omega$ is a nontrivial cube root of unity. This is no longer true if we replace $\mathbf{Q}$ by $\mathbf{R}$, since $T^3 - 2$ has one root in $\mathbf{R}$.

**Theorem 2.3.** *The roots of $T^n - a$ in a splitting field over $K$ are numbers of the form $\zeta \sqrt[n]{a}$ where $\zeta$ is an $n$th root of unity ($\zeta^n = 1$) in $K$.*

*Proof.* Set $\alpha = \sqrt[n]{a}$, which is a fixed choice of root of $T^n - a$ over $K$. If $\beta$ is another root in a splitting field of $T^n - a$ over $K$ then $\beta^n = a = \alpha^n$, so $(\beta/\alpha)^n = 1$. Set $\zeta = \beta/\alpha \in K$, so $\beta = \zeta\alpha = \zeta \sqrt[n]{a}$ and $\zeta^n = (\beta/\alpha)^n = 1$.

Conversely, if $\zeta^n = 1$ and $\zeta \in K$ then $(\zeta \sqrt[n]{a})^n = \zeta^n a = a$, so $\zeta \sqrt[n]{a}$ is a root of $T^n - a$ in $K$. □

## 3. Prime exponents

In degree greater than 3, lack of roots ordinarily does *not* imply irreducibility. Consider $(T^2-2)(T^2-3)$ in $\mathbf{Q}[T]$. The polynomial $T^p - a$, where the exponent is prime, is a surprising counterexample: for these polynomials lack of a root is equivalent to irreducibility.

**Theorem 3.1.** *For an arbitrary field $K$ and prime number $p$, and $a \in K^\times$, $T^p - a$ is irreducible in $K[T]$ if and only if it has no root in $K$. Equivalently, $T^p - a$ is reducible in $K[T]$ if and only if it has a root in $K$.*

*Proof.* Clearly if $T^p - a$ is irreducible in $K[T]$ then it has no root in $K$ (since its degree is greater than 1).

In order to prove that $T^p - a$ not having a root in $K$ implies it is irreducible we will prove the contrapositive: if $T^p - a$ is reducible in $K[T]$ then it has a root in $K$.

Write $T^p - a = g(T)h(T)$ in $K[T]$ where $m = \deg g$ satisfies $1 \leq m \leq p - 1$. Since $T^p - a$ is monic the leading coefficients of $g$ and $h$ multiply to 1, so by rescaling (which doesn't change degrees) we may assume $g$ is monic and thus $h$ is monic.

Let $L$ be a splitting field of $T^p - a$ over $K$ and $\alpha = \sqrt[p]{a}$ be one root of $T^p - a$ in $L$. Its other roots in $L$ are $\zeta\alpha$ where $\zeta^p = 1$ (Theorem 2.3), so in $L[T]$

$$T^p - a = (T - \zeta_1\alpha)(T - \zeta_2\alpha) \cdots (T - \zeta_p\alpha)$$

where $\zeta_i^p = 1$. (Possibly $\zeta_i = \zeta_j$ when $i \neq j$; whether or not this happens doesn't matter.) By unique factorization in $L[T]$, every monic factor of $T^p - a$ in $L[T]$ is a product of some number of $(T - \zeta_i\alpha)$'s. Therefore

$$g(T) = (T - \zeta_{i_1}\alpha)(T - \zeta_{i_2}\alpha) \cdots (T - \zeta_{i_m}\alpha) \tag{3.1}$$

for some $p$th roots of unity $\zeta_{i_1}, \ldots, \zeta_{i_m}$.

Now let's *look at the constant terms* in (3.1). Set $c = g(0)$, so

$$c = (-1)^m(\zeta_{i_1} \cdots \zeta_{i_m})\alpha^m.$$

Since $g(T) \in K[T]$, $c \in K$ and $c \neq 0$ on account of $g(0)h(0) = 0^p - a = -a$. Therefore

$$c = (-1)^m(\zeta_{i_1} \cdots \zeta_{i_m})\alpha^m \in K^\times. \tag{3.2}$$

We want to replace $\alpha^m$ with $\alpha$, and will do this by raising $\alpha^m$ to an additional power to make the exponent on $\alpha$ congruent to 1 mod $p$.

Because $p$ is prime and $1 \leq m \leq p-1$, $m$ and $p$ are relatively prime: we can write $mx + py = 1$ for some $x$ and $y$ in $\mathbf{Z}$. Raise the product in (3.2) to the $x$-power to make the exponent on $\alpha$ equal to $mx = 1 - py$:

$$\begin{aligned} c^x &= (-1)^{mx}(\zeta_{i_1} \cdots \zeta_{i_m})^x \alpha^{mx} \\ &= (-1)^{mx}(\zeta_{i_1} \cdots \zeta_{i_m})^x \alpha^{1-py} \\ &= (-1)^{mx}(\zeta_{i_1} \cdots \zeta_{i_m})^x \frac{\alpha}{(\alpha^p)^y} \\ &= (-1)^{mx}(\zeta_{i_1} \cdots \zeta_{i_m})^x \frac{\alpha}{a^y}, \end{aligned}$$

so

$$(\zeta_{i_1} \cdots \zeta_{i_m})^x \alpha = a^y(-1)^{mx}c^x \in K^\times$$

and the left side has the form $\zeta\alpha$ where $\zeta^p = 1$, so $K$ contains a root of $T^p - a$. □

**Remark 3.2.** For an odd prime $p$ and any field $K$, the irreducibility of $T^p - a$ over $K$ implies irreducibility of $T^{p^r} - a$ for all $r \geq 1$, which is not obvious! And this doesn't quite work when $p = 2$: irreducibility of $T^4 - a$ implies irreducibility of $T^{2^r} - a$ for all $r \geq 2$ (again, not obvious!), but irreducibility of $T^2 - a$ need not imply irreducibility of $T^4 - a$. A basic example is that $T^2 + 4$ is irreducible in $\mathbf{Q}[T]$ but $T^4 + 4 = (T^2 + 2T + 2)(T^2 - 2T + 2)$. See [2, pp. 297–298] for a precise irreducibility criterion for $T^n - a$ over any field, which is due to Vahlen [4] in 1895 for $K = \mathbf{Q}$, Capelli [1] in 1897 for $K$ of characteristic 0, and Rédei [3] in 1959 for positive characteristic.

## 4. Irreducibility relations among $T^n - a$ for different exponents

**Theorem 4.1.** *Let $K$ be a field, $a \in K^\times$, and assume $T^n - a$ is irreducible over $K$. If $d \mid n$ then $T^d - a$ is irreducible over $K$. Equivalently, if $[K(\sqrt[n]{a}) : K] = n$ for some $n$th root of $a$ over $K$ then for all $d \mid n$ we have $[K(\sqrt[d]{a}) : K] = d$ for every $d$th root of $a$.*

*Proof.* We prove irreducibility of $T^n - a$ implies irreducibility of $T^d - a$ in two ways: working with polynomials and working with field extensions.

Polynomials: assume $T^d - a$ is reducible over $K$, so $T^d - a = g(T)h(T)$ where $0 < \deg g(T) < d$. Replacing $T$ with $T^{n/d}$ in this equation, we get $T^n - a = g(T^{n/d})h(T^{n/d})$ where $\deg g(T^{n/d}) = (n/d)\deg g < (n/d)d = n$ and clearly $\deg g(T^{n/d}) > 0$.

Field extensions: let $\sqrt[n]{a}$ be an $n$th root of $a$ over $K$, so $[K(\sqrt[n]{a}) : K] = n$ by Theorem 2.1. Define $\sqrt[d]{a} = \sqrt[n]{a}^{\,n/d}$. This is a root of $T^d - a$ since $\sqrt[d]{a}^{\,d} = (\sqrt[n]{a}^{\,n/d})^d = \sqrt[n]{a}^{\,n} = a$. To prove $T^d - a$ is irreducible over $K$ we will prove $[K(\sqrt[d]{a}) : K] = d$ using that choice of $\sqrt[d]{a}$.

In the tower $K \subset K(\sqrt[d]{a}) \subset K(\sqrt[n]{a})$, we have $[K(\sqrt[d]{a}) : K] \leq d$ and $[K(\sqrt[n]{a}) : K(\sqrt[d]{a})] \leq n/d$ by Theorem 2.1, since $\sqrt[d]{a}$ is a root of $T^d - a \in K[T]$ and $\sqrt[n]{a}$ is a root of $T^{n/d} - \sqrt[d]{a} \in K(\sqrt[d]{a})[T]$. We have

$$[K(\sqrt[n]{a}) : K] = [K(\sqrt[n]{a}) : K(\sqrt[d]{a})][K(\sqrt[d]{a}) : K]$$

and our irreducibility hypothesis implies the left side is $n$, so it follows that our upper bounds $n/d$ and $d$ for the factors on the right must be equalities. In particular, $[K(\sqrt[d]{a}) : K] = d$ so $T^d - a$ is irreducible over $K$ (it has a root with degree $d$ over $K$). □

There was an important calculation in this proof that we will use repeatedly below: if $d \mid n$ then $K(\sqrt[n]{a})$ contains $K(\sqrt[d]{a})$, where $\sqrt[d]{a} := \sqrt[n]{a}^{n/d}$. This is a root of $T^d - a$, so the notation is reasonable, but note that $\sqrt[d]{a}$ is not an arbitrary $d$th root of $a$: it depends on the choice made first of $\sqrt[n]{a}$.

By Theorem 4.1 and Remark 3.2, for odd primes $p$ irreducibility of $T^p - a$ is equivalent to irreducibility of $T^{p^r} - a$ for any single $r \geq 1$, and for the prime 2 irreducibility of $T^4 - a$ is equivalent to irreducibility of $T^{2^r} - a$ for any single $r \geq 2$.

**Theorem 4.2.** *For relatively prime positive integers $m$ and $n$, $T^{mn} - a$ is irreducible over $K$ if and only if $T^m - a$ and $T^n - a$ are each irreducible over $K$. Equivalently, if $m$ and $n$ are relatively prime positive integers then $[K(\sqrt[mn]{a}) : K] = mn$ if and only if $[K(\sqrt[m]{a}) : K] = m$ and $[K(\sqrt[n]{a}) : K] = n$.*

*Proof.* That irreducibility of $T^{mn} - a$ over $K$ implies irreducibility of $T^m - a$ and $T^n - a$ over $K$ follows from Theorem 4.1.

To prove irreducibility of $T^m - a$ and $T^n - a$ over $K$ implies irreducibility of $T^{mn} - a$ over $K$ we will work with roots of these polynomials. It is convenient to select $m$th, $n$th, and $mn$th roots of $a$ in a multiplicatively compatible way: fix a root $\sqrt[mn]{a}$ of $T^{mn} - a$ over $K$ and define $\sqrt[m]{a} := \sqrt[mn]{a}^n$ and $\sqrt[n]{a} := \sqrt[mn]{a}^m$. Then $\sqrt[m]{a}$ is a root of $T^m - a$ and $\sqrt[n]{a}$ is a root of $T^n - a$, so we have the following field diagram, where the containments are due to $\sqrt[m]{a}$ and $\sqrt[n]{a}$ being powers of $\sqrt[mn]{a}$.

$$\begin{tikzcd}[column sep=small]
 & K(\sqrt[mn]{a}) \arrow[dl, dash, "\leq n"'] \arrow[dr, dash, "\leq m"] & \\
K(\sqrt[m]{a}) \arrow[dr, dash, "m"'] & & K(\sqrt[n]{a}) \arrow[dl, dash, "n"] \\
 & K &
\end{tikzcd}$$

The bottom field degree values come from $T^m - a$ and $T^n - a$ being irreducible over $K$, and the top field degree upper bounds come from $\sqrt[mn]{a}$ being a root of $T^n - \sqrt[m]{a} \in K(\sqrt[m]{a})[T]$ and $T^m - \sqrt[n]{a} \in K(\sqrt[n]{a})[T]$. Let $d = [K(\sqrt[mn]{a}) : K]$, so by reading the field diagram along either the left or right we have $\boxed{d \leq mn}$. Also $d$ is divisible by $m$ and by $n$ since field degrees are multiplicative in towers, so from relative primality of $m$ and $n$ we get $m \mid d, n \mid d \Longrightarrow mn \mid d$, so $\boxed{mn \leq d}$. Thus $d = mn$, so $T^{mn} - a$ is the minimal polynomial of $\sqrt[mn]{a}$ over $K$ and thus is irreducible over $K$. $\square$

**Corollary 4.3.** *For an integer $N > 1$ with prime factorization $p_1^{e_1} \cdots p_k^{e_k}$, $T^N - a$ is irreducible over $K$ if and only if each $T^{p_i^{e_i}} - a$ is irreducible over $K$.*

*Proof.* Use Theorem 4.2 with the factorization $N = p_1^{e_1}(p_2^{e_2} \cdots p_k^{e_k})$ to see irreducibility of $T^N - a$ over $K$ is equivalent to irreducibility of $T^{p_1^{e_1}} - a$ and $T^{p_2^{e_2} \cdots p_k^{e_k}} - a$ over $K$, and then by induction on the number of different prime powers in the degree, irreducibility of $T^{p_2^{e_2} \cdots p_k^{e_k}} - a$ over $K$ is equivalent to irreducibility of $T^{p_i^{e_i}} - a$ over $K$ for $i = 2, \ldots, k$. $\square$

**Example 4.4.** Irreducibility of $T^{90} - a$ over $K$ is equivalent to irreducibility of $T^2 - a$, $T^9 - a$, and $T^5 - a$ over $K$.

**Remark 4.5.** By Remark 3.2, if $N$ is odd then irreducibility of $T^N - a$ over $K$ is equivalent to irreducibility of $T^{p_i} - a$ over $K$ as $p_i$ runs over the prime factors of $N$ (the multiplicities $e_i$ don't matter!), and for these we know the story for irreducibility by Theorem 3.1: it's the same thing as $T^{p_i} - a$ not having a root in $K$ for each $p_i$.

**Example 4.6.** Irreducibility of $T^{75} - a$ over $K$ is equivalent to $a$ not having a cube root or fifth root in $K$.

## 5. Intermediate fields in a simple radical extension

For a choice of $n$th root $\sqrt[n]{a}$ and a factor $d \mid n$, $\sqrt[d]{a} := \sqrt[n]{a}^{\,n/d}$ is a root of $T^d - a$ in $K(\sqrt[n]{a})$, so we have the following field diagram.

$$\begin{tikzcd} K(\sqrt[n]{a}) \arrow[d, dash] \\ K(\sqrt[d]{a}) \arrow[d, dash] \\ K \end{tikzcd}$$

It's natural to ask if every field between $K$ and $K(\sqrt[n]{a})$ is $K(\sqrt[d]{a})$ for some $d$ dividing $n$. The simplest setting to study this is when $T^n - a$ is irreducible over $K$ (and thus also $T^d - a$ is irreducible over $K$, by Theorem 4.1), so $[K(\sqrt[d]{a}) : K] = d$. Is $K(\sqrt[d]{a})$ the only extension of $K$ of degree $d$ inside $K(\sqrt[n]{a})$? This is not always true.

**Example 5.1.** Let $K = \mathbf{Q}$ and consider the field $\mathbf{Q}(\sqrt[4]{-1})$. Set $\alpha = \sqrt[4]{-1}$, so $\alpha^4 + 1 = 0$. The polynomial $T^4 + 1$ is irreducible over $\mathbf{Q}$ because it becomes Eisenstein at 2 when $T$ is replaced with $T + 1$. Since $[\mathbf{Q}(\sqrt[4]{-1}) : \mathbf{Q}] = 4$, any field strictly between $\mathbf{Q}$ and $\mathbf{Q}(\sqrt[4]{-1})$ is quadratic over $\mathbf{Q}$. One of these is $\mathbf{Q}(\sqrt{-1})$, but it is not the only one.

$$\begin{tikzcd} & \mathbf{Q}(\sqrt[4]{-1}) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\ \mathbf{Q}(\sqrt{2}) \arrow[dr, dash] & \mathbf{Q}(\sqrt{-1}) \arrow[d, dash] & \mathbf{Q}(\sqrt{-2}) \arrow[dl, dash] \\ & \mathbf{Q} & \end{tikzcd}$$

If $\alpha^4 = -1$ then $(\alpha + 1/\alpha)^2 = \alpha^2 + 2 + 1/\alpha^2 = (\alpha^4 + 1)/\alpha^2 + 2 = 2$ and $(\alpha - 1/\alpha)^2 = \alpha^2 - 2 + 1/\alpha^2 = (\alpha^4 + 1)/\alpha^2 - 2 = -2$, so $\mathbf{Q}(\sqrt[4]{-1})$ contains $\mathbf{Q}(\sqrt{2})$ and $\mathbf{Q}(\sqrt{-2})$. None of the fields $\mathbf{Q}(i)$, $\mathbf{Q}(\sqrt{2})$, and $\mathbf{Q}(\sqrt{-2})$ are the same, so we have at least three (and in fact there are just these three) quadratic extensions of $\mathbf{Q}$ in $\mathbf{Q}(\sqrt[4]{-1})$.

In the above example, the "reason" for the appearance of more intermediate fields between $\mathbf{Q}$ and $\mathbf{Q}(\sqrt[4]{-1})$ than just $\mathbf{Q}(\sqrt{-1})$ is that there are 4th roots of unity in $\mathbf{Q}(\sqrt[4]{-1})$ that are not in $\mathbf{Q}$, namely $\pm\sqrt{-1}$. The following theorem shows we get no such unexpected fields if all $n$th roots of unity in the top field are actually in the base field.

**Theorem 5.2.** *Let $K$ be a field, $a \in K^\times$, and assume $T^n - a$ is irreducible over $K$. If all $n$th roots of unity in $K(\sqrt[n]{a})$ are in $K$ then for each $d \mid n$ the only field between $K$ and $K(\sqrt[n]{a})$ of degree $d$ over $K$ is $K(\sqrt[d]{a})$, where $\sqrt[d]{a} := \sqrt[n]{a}^{n/d}$.*

*Proof.* Every field between $K$ and $K(\sqrt[n]{a})$ has degree over $K$ that divides $n$. For $d \mid n$ suppose $L$ is a field with $K \subset L \subset K(\sqrt[n]{a})$ and $[L : K] = d$. To prove $L = K(\sqrt[d]{a})$, it suffices to show $\sqrt[d]{a} \in L$, since that would give us $K(\sqrt[d]{a}) \subset L$ and we know $K(\sqrt[d]{a})$ has degree $d$ over $K$, so the containment $K(\sqrt[d]{a}) \subset L$ would have to be an equality.

$$\begin{array}{c} K(\sqrt[n]{a}) \\ \Big| \, {\scriptstyle n/d} \\ L \\ \Big| \, {\scriptstyle d} \\ K \end{array}$$

Let $f(T)$ be the minimal polynomial of $\sqrt[n]{a}$ **over** $L$, so $f(T) \mid (T^n - a)$ and $\deg f = n/d$. We can write any other root of $f(T)$ as $\zeta \sqrt[n]{a}$ for some $n$th root of unity $\zeta$. (Theorem 2.3). In a splitting field of $T^n - a$ over $K$, the factorization of $f(T)$ is $\prod_{i \in I}(T - \zeta_i \sqrt[n]{a})$ for some $n$th roots of unity $\zeta_i$ ($I$ is just an index set). The constant term of $f(T)$ is in $L$, so $(\prod_{i \in I} \zeta_i) \sqrt[n]{a}^{n/d} \in L$. Therefore $(\prod_{i \in I} \zeta_i) \sqrt[n]{a}^{n/d} \in K(\sqrt[n]{a})$, so $\prod_{i \in I} \zeta_i \in K(\sqrt[n]{a})$. The only $n$th roots of unity in $K(\sqrt[n]{a})$ are, by hypothesis, in $K$, so $\prod_{i \in I} \zeta_i \in K \subset L$. Therefore $\sqrt[n]{a}^{n/d} = \sqrt[d]{a}$ is in $L$, so we're done. $\square$

**Example 5.3.** If $K = \mathbf{Q}$, $a > 0$, and $T^n - a$ is irreducible over $\mathbf{Q}$ then $\mathbf{Q}(\sqrt[n]{a})$ is isomorphic to a subfield of $\mathbf{R}$ (using the *real* positive $n$th root of $a$), which implies the only roots of unity in $\mathbf{Q}(\sqrt[n]{a})$ are $\pm 1$ and those both lie $\mathbf{Q}$. For example, the only fields between $\mathbf{Q}$ and $\mathbf{Q}(\sqrt[n]{2})$ are $\mathbf{Q}(\sqrt[d]{2})$ where $d \mid n$ and $\sqrt[d]{2} = \sqrt[n]{2}^{n/d}$.

**Example 5.4.** Let $F$ be a field and $K = F(u)$, the rational functions over $F$ in one indeterminate. The polynomial $T^n - u$ is irreducible over $F(u)$ since it is Eisenstein at $u$. We let $\sqrt[n]{u}$ denote one root of $T^n - u$, so $K(\sqrt[n]{u}) = F(\sqrt[n]{u})$ has degree $n$ over $F(u)$. All roots of unity in $F(\sqrt[n]{u})$ – not just $n$th roots of unity – are in $F$, because $F(\sqrt[n]{u})$ is itself a rational function field in one indeterminate over $F$ (since $\sqrt[n]{u}$ is transcendental over $F$) and all elements of a rational function field in one indeterminate over $F$ that are not in $F$ are transcendental over $F$ and thus can't be a root of unity. Therefore by Theorem 5.2, the fields between $F(u)$ and $F(\sqrt[n]{u})$ are $F(\sqrt[d]{u})$ for $d \mid n$.

**Example 5.5.** An example where the hypothesis that all $n$th roots of unity in $K(\sqrt[n]{a})$ are in $K$ is false, yet the conclusion of Theorem 5.2 is true, is $K = \mathbf{Q}(i)$, $a = 2$, and $n = 8$: it can be shown that $[\mathbf{Q}(i, \sqrt[8]{2}) : \mathbf{Q}(i)] = 8$ and the only fields between $\mathbf{Q}(i)$ and $\mathbf{Q}(i, \sqrt[8]{2})$ are $\mathbf{Q}(i, \sqrt[d]{2})$ for $d = 1, 2, 4, 8$ while $\frac{1+i}{\sqrt{2}}$ is an 8th root of unity in $\mathbf{Q}(\sqrt[8]{2}, i)$ that is not in $\mathbf{Q}(i)$.

## References

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