

# CYCLOTOMIC EXTENSIONS

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## 1. INTRODUCTION

For a positive integer  $n$ , an  $n$ th root of unity in a field is a solution to  $z^n = 1$ , or equivalently is a root of  $T^n - 1$ . There are at most  $n$  different  $n$ th roots of unity in a field since  $T^n - 1$  has at most  $n$  roots in any field. A root of unity is an  $n$ th root of unity for some  $n$ . The only roots of unity in  $\mathbf{R}$  are  $\pm 1$ , while in  $\mathbf{C}$  there are  $n$  different  $n$ th roots of unity for each  $n$ , namely  $e^{2\pi ik/n}$  for  $0 \leq k \leq n - 1$  and they form a group of order  $n$ . In characteristic  $p$  there is no  $p$ th root of unity besides 1: if  $x^p = 1$  in characteristic  $p$  then  $0 = x^p - 1 = (x - 1)^p$ , so  $x = 1$ . That is strange, but it is a key feature of characteristic  $p$ , *e.g.*, it makes the  $p$ th power map  $x \mapsto x^p$  on fields of characteristic  $p$  injective.

For any field  $K$ , an extension of the form  $K(\zeta)$ , where  $\zeta$  is a root of unity, is called a *cyclotomic* extension of  $K$ . The term cyclotomic means “circle-dividing,” which comes from the fact that the  $n$ th roots of unity in  $\mathbf{C}$  divide a circle into  $n$  arcs of equal length, as in Figure 1 when  $n = 7$ . The important algebraic fact we will explore is that cyclotomic extensions of every field have an abelian Galois group; we will look especially at cyclotomic extensions of  $\mathbf{Q}$  and finite fields. There are not many general methods known for constructing abelian extensions (that is, Galois extensions with abelian Galois group); cyclotomic extensions are essentially the only construction that works over all fields. Other constructions of abelian extensions are Kummer extensions, Artin-Schreier-Witt extensions, and Carlitz extensions, but these all require special conditions on the base field.

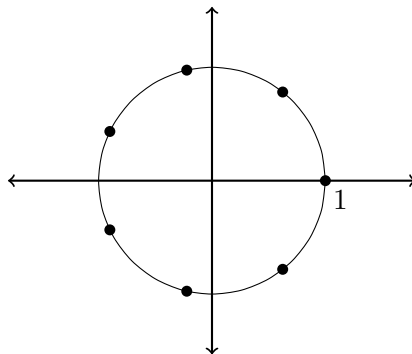


FIGURE 1. The 7th roots of unity.

The  $n$ th roots of unity in any field form a group under multiplication. It’s obvious that this group in  $\mathbf{C}$  is cyclic from the analytic formula for them, with generator  $e^{2\pi i/n}$ . In a general field there is no formula, but these roots of unity are still a cyclic group.

**Theorem 1.1.** *The group of  $n$ th roots of unity in any field is cyclic. More generally, any finite subgroup of the nonzero elements of a field is a cyclic group.*

*Proof.* Let  $F$  be a field and  $G$  be a finite subgroup of  $F^\times$ . From the general theory of abelian groups, if there are elements in  $G$  with orders  $n_1$  and  $n_2$  then there is an element of  $G$  with order the least common multiple  $[n_1, n_2]$ . Letting  $N$  be the maximal order of all the elements of  $G$ , we will show the order of every element in  $G$  divides  $N$ . If  $n$  is the order of any element in  $G$  then there is an element of  $G$  with order  $[n, N] \geq N$ . Since  $N$  is the maximal order we have  $[n, N] \leq N$ , so  $[n, N] = N$ , which implies  $n$  divides  $N$  (why?). Thus every element of  $G$  is a root of  $T^N - 1$ , which implies  $\#G \leq N$  (the number of roots of a polynomial in a field is at most its degree). At the same time, since the order of each element divides the size of the group we have  $N \mid \#G$ . Hence  $N = \#G$ , which means some element of  $G$  has order  $\#G$ , so  $G$  is cyclic.  $\square$

**Example 1.2.** For any prime  $p$ , the group  $(\mathbf{Z}/(p))^\times$  is cyclic by Theorem 1.1 since these are the nonzero elements in the field  $\mathbf{Z}/(p)$  and they form a finite group. More generally, if  $F$  is any finite field then  $F^\times$  is a cyclic group.

**Watch out!** Theorem 1.1 does *not* say  $(\mathbf{Z}/(p^r))^\times$  is cyclic for  $r > 1$ , since the ring  $\mathbf{Z}/(p^r)$  is not a field for  $r > 1$ . The theorem is simply silent about this. In fact, for other reasons,  $(\mathbf{Z}/(p^r))^\times$  is cyclic for  $p \neq 2$  but usually not if  $p = 2$ , *e.g.*,  $(\mathbf{Z}/(8))^\times$  is not cyclic.

For a cyclotomic extension  $K(\zeta)/K$  set  $n$  to be the order of  $\zeta$  as a root of unity: the least exponent making  $\zeta^n = 1$ . Then  $T^n - 1$  has every power of  $\zeta$  as a root, so it has  $n$  different roots:  $T^n - 1$  is *separable* over  $K$ . Conversely, if  $T^n - 1$  is separable over  $K$  then it has  $n$  different roots, they are a group under multiplication, and Theorem 1.1 guarantees it is cyclic: there is a root of unity of order  $n$  among the  $n$ th roots of unity. Therefore when we construct cyclotomic extensions  $K(\zeta)/K$  little is lost by assuming  $T^n - 1$  is separable over  $K$ . That is equivalent to  $T^n - 1$  being relatively prime to its derivative  $nT^{n-1}$  in  $K[T]$ , which is equivalent to  $n \neq 0$  in  $K$ :  $\text{char}(K) = 0$ , or  $\text{char}(K) = p$  and  $(p, n) = 1$ . We assume this is the case in all we do below.

When there are  $n$  different  $n$ th roots of unity, we denote the group of them by  $\mu_n$ .<sup>1</sup> For instance, in  $\mathbf{C}$  we have  $\mu_2 = \{1, -1\}$  and  $\mu_4 = \{1, -1, i, -i\}$ . In  $\mathbf{F}_7$ ,  $\mu_3 = \{1, 2, 4\}$ . A generator of  $\mu_n$  is denoted  $\zeta_n$ . That is,  $\zeta_n$  denotes a root of unity of order  $n$ . **Watch out!** An  $n$ th root of unity is a solution to  $z^n = 1$  but that doesn't mean it has order  $n$ . For example, 1 is an  $n$ th root of unity for every  $n \geq 1$ . An  $n$ th root of unity that has order  $n$  is called a *primitive*  $n$ th roots of unity ( $z^n = 1$  and  $z^j \neq 1$  for  $j < n$ ). For example,  $-1$  in  $\mathbf{C}$  is a 4th root of unity but not a primitive 4th root of unity. For  $a \in \mathbf{Z}$ , the order of  $\zeta_n^a$  is  $n/(a, n)$ , so  $\zeta_n^a$  is a primitive  $n$ th root of unity if and only if  $(a, n) = 1$ . Therefore when a field contains  $n$  different  $n$ th roots of unity it contains  $\varphi(n)$  primitive  $n$ th roots of unity, where  $\varphi(n) = \#(\mathbf{Z}/(n))^\times$ . The primitive  $n$ th roots of unity are the generators of  $\mu_n$ , and when  $n \geq 3$  there is not a unique generator *e.g.*, if  $\zeta_n$  is one generator then  $\zeta_n^{-1}$  is another one), so writing  $\zeta_n$  always involves making a *choice* of generator.

Any two primitive  $n$ th root of unity in a field are powers of each other, so the extension  $K(\zeta_n)$  is independent of the choice of  $\zeta_n$ . We will often write this field as  $K(\mu_n)$ : adjoining one primitive  $n$ th root of unity is the same as adjoining a full set of  $n$ th roots of unity.

## 2. EMBEDDING THE GALOIS GROUP

When  $T^n - 1$  is separable over  $K$ ,  $K(\mu_n)/K$  is Galois since  $K(\mu_n)$  is the splitting field of  $T^n - 1$  over  $K$ . What is its Galois group?

<sup>1</sup>Anytime we write  $\mu_n$  it is understood to contain  $n$  elements.

**Lemma 2.1.** For  $\sigma \in \text{Gal}(K(\mu_n)/K)$  there is an integer  $a = a_\sigma$  that is relatively prime to  $n$  such that  $\sigma(\zeta) = \zeta^a$  for all  $\zeta \in \mu_n$ .

*Proof.* Let  $\zeta_n$  be a generator of  $\mu_n$  (that is, a primitive  $n$ th root of unity), so  $\zeta_n^n = 1$  and  $\zeta_n^j \neq 1$  for  $1 \leq j < n$ . Then  $\sigma(\zeta_n)^n = 1$  and  $\sigma(\zeta_n)^j \neq 1$  for  $1 \leq j < n$ , so  $\sigma(\zeta_n)$  is a primitive  $n$ th root of unity. This implies  $\sigma(\zeta_n) = \zeta_n^a$  where  $(a, n) = 1$ . Any  $\zeta \in \mu_n$  has the form  $\zeta_n^k$  for some  $k$ , so

$$\sigma(\zeta) = \sigma(\zeta_n^k) = \sigma(\zeta_n)^k = (\zeta_n^a)^k = (\zeta_n^k)^a = \zeta^a.$$

□

The exponent  $a$  in Lemma 2.1 is well-defined modulo  $n$ :  $\zeta_n^a = \zeta_n^b \Rightarrow a \equiv b \pmod n$  because  $\zeta_n$  has order  $n$ . Thus we can think of  $a = a_\sigma$  as an element of the group  $(\mathbf{Z}/(n))^\times$ .

**Example 2.2.** The primitive 7th roots of unity are the 7th roots of unity besides 1, and they are all roots of  $(T^7 - 1)/(T - 1) = T^6 + T^5 + T^4 + T^3 + T^2 + T + 1$ . This polynomial is irreducible over  $\mathbf{Q}$  because it becomes Eisenstein at 7 when we replace  $T$  with  $T + 1$ :

$$\frac{(T+1)^7 - 1}{(T+1) - 1} = T^6 + 7T^5 + 21T^4 + 35T^3 + 35T^2 + 21T + 7.$$

This implies, for instance, that  $\zeta_7$  and  $\zeta_7^2$  have the same minimal polynomial over  $\mathbf{Q}$ . Since, moreover,  $\mathbf{Q}(\zeta_7) = \mathbf{Q}(\zeta_7^2)$ , there is an automorphism  $\sigma \in \text{Gal}(\mathbf{Q}(\mu_7)/\mathbf{Q})$  with the effect  $\sigma(\zeta) = \zeta^2$  for all  $\zeta \in \mu_7$ . It is **FALSE** that  $\sigma(\alpha) = \alpha^2$  for all  $\alpha \in \mathbf{Q}(\mu_7)$ , since squaring is not additive in characteristic 0. **ONLY** on the 7th roots of unity is  $\sigma$  being described as a power map. Elsewhere  $\sigma$  is determined from additivity and multiplicativity, *e.g.*,

$$\sigma(4\zeta_7^5 - 11\zeta_7 + 9) = 4(\zeta_7^2)^5 - 11\zeta_7^2 + 9 = 4\zeta_7^3 - 11\zeta_7^2 + 9.$$

**Theorem 2.3.** *The mapping*

$$\text{Gal}(K(\mu_n)/K) \rightarrow (\mathbf{Z}/(n))^\times$$

where  $\sigma \mapsto a_\sigma \pmod n$ , from  $\sigma(\zeta) = \zeta^{a_\sigma}$  for all  $\zeta \in \mu_n$ , is an injective group homomorphism.

*Proof.* Pick  $\sigma$  and  $\tau$  in  $\text{Gal}(K(\mu_n)/K)$ . For a primitive  $n$ th root of unity  $\zeta_n$ ,

$$(\sigma\tau)(\zeta_n) = \sigma(\tau(\zeta_n)) = \sigma(\zeta_n^{a_\tau}) = \sigma(\zeta_n)^{a_\tau} = (\zeta_n^{a_\sigma})^{a_\tau} = \zeta_n^{a_\sigma a_\tau}.$$

Also  $(\sigma\tau)(\zeta_n) = \zeta_n^{a_{\sigma\tau}}$ , so  $\zeta_n^{a_{\sigma\tau}} = \zeta_n^{a_\sigma a_\tau}$ . Since  $\zeta_n$  has order  $n$ ,  $a_{\sigma\tau} \equiv a_\sigma a_\tau \pmod n$ . This shows  $\sigma \mapsto a_\sigma \pmod n$  is a homomorphism from  $\text{Gal}(K(\mu_n)/K)$  to  $(\mathbf{Z}/(n))^\times$ .

When  $\sigma$  is in the kernel,  $a_\sigma \equiv 1 \pmod n$ , so  $\sigma(\zeta_n) = \zeta_n$ . Also  $\sigma$  fixes all the elements of  $K$ , so  $\sigma$  is the identity on  $K(\zeta_n) = K(\mu_n)$ , so  $\sigma$  is the identity in  $\text{Gal}(K(\mu_n)/K)$ . □

Since  $(\mathbf{Z}/(n))^\times$  is abelian, the embedded subgroup  $\text{Gal}(K(\mu_n)/K)$  is abelian. We have proved that *cyclotomic extensions are always abelian*. Whenever we view  $\text{Gal}(K(\mu_n)/K)$  in  $(\mathbf{Z}/(n))^\times$ , it will always be understood to be by the embedding in Theorem 2.3.

**Example 2.4.** Complex conjugation is an automorphism of  $\mathbf{Q}(\mu_n)/\mathbf{Q}$  with order 2. Under the embedding of  $\text{Gal}(\mathbf{Q}(\mu_n)/\mathbf{Q})$  into  $(\mathbf{Z}/(n))^\times$ , complex conjugation corresponds to  $-1 \pmod n$  since  $\bar{\zeta} = \zeta^{-1}$  for any root of unity  $\zeta$ .

The embedding of  $\text{Gal}(K(\mu_n)/K)$  into  $(\mathbf{Z}/(n))^\times$  may not be surjective; that depends on  $K$ . For instance, if  $K = \mathbf{R}$  and  $n \geq 3$  then  $K(\mu_n)/K = \mathbf{C}/\mathbf{R}$  is a quadratic extension. The nontrivial  $\mathbf{R}$ -automorphism of  $\mathbf{C}$  is complex conjugation, whose effect on roots of unity in  $\mathbf{C}$  is to invert them:  $\bar{\zeta} = \zeta^{-1}$ . Therefore the embedding  $\text{Gal}(\mathbf{C}/\mathbf{R}) \hookrightarrow (\mathbf{Z}/(n))^\times$  for  $n \geq 3$  has image  $\{\pm 1 \pmod n\}$ , which is smaller than  $(\mathbf{Z}/(n))^\times$  unless  $n = 2, 3, 4$ , or  $6$ .

We will figure out the image in Theorem 2.3 in two important examples:  $K = \mathbf{Q}$  and  $K = \mathbf{F}_p$ . In the first case the embedding  $\text{Gal}(K(\mu_n)/K) \rightarrow (\mathbf{Z}/(n))^\times$  is surjective, while in the second case the embedding need not be surjective but we can still describe the image.

**Theorem 2.5.** *The embedding  $\text{Gal}(\mathbf{Q}(\mu_n)/\mathbf{Q}) \hookrightarrow (\mathbf{Z}/(n))^\times$  is an isomorphism.*

*Proof.* Let  $\zeta_n$  be a primitive  $n$ th root of unity over  $\mathbf{Q}$ . The size of  $\text{Gal}(\mathbf{Q}(\mu_n)/\mathbf{Q}) = \text{Gal}(\mathbf{Q}(\zeta_n)/\mathbf{Q})$  is the number of  $\mathbf{Q}$ -conjugates of  $\zeta_n$ , which is at most  $\varphi(n) = \#(\mathbf{Z}/(n))^\times$ . To prove  $\text{Gal}(\mathbf{Q}(\mu_n)/\mathbf{Q}) \hookrightarrow (\mathbf{Z}/(n))^\times$  is a surjection we will show for all  $a \in \mathbf{Z}$  such that  $(a, n) = 1$  that  $\zeta_n$  and  $\zeta_n^a$  are  $\mathbf{Q}$ -conjugate: their minimal polynomials over  $\mathbf{Q}$  agree.

Since  $\zeta_n^a$  only depends on  $a \bmod n$ , we can take  $a > 0$ , and in fact  $a > 1$ . Write  $a = p_1 p_2 \cdots p_r$  as a product of primes  $p_i$ , each not dividing  $n$  (some  $p_i$ 's could coincide). To show  $\zeta_n$  and  $\zeta_n^a$  have the same minimal polynomial over  $\mathbf{Q}$ , it suffices to show for each prime  $p$  not dividing  $n$  that any primitive  $n$ th root of unity and its  $p$ th power have the same minimal polynomial over  $\mathbf{Q}$ , since then the successive pairs of primitive  $n$ th roots of unity

$$\zeta_n, \zeta_n^{p_1}, \zeta_n^{p_1 p_2}, \zeta_n^{p_1 p_2 p_3}, \dots, \zeta_n^{p_1 p_2 \cdots p_r} = \zeta_n^a$$

have the same minimal polynomial over  $\mathbf{Q}$  since each is a prime power of the previous one.

For an arbitrary primitive  $n$ th root of unity  $\zeta_n$  over  $\mathbf{Q}$ , assume  $\zeta_n$  and  $\zeta_n^p$  are *not*  $\mathbf{Q}$ -conjugate for some prime  $p$  not dividing  $n$ . We will get a contradiction. Let  $f(T)$  be the minimal polynomial of  $\zeta_n$  over  $\mathbf{Q}$  and  $g(T)$  be the minimal polynomial of  $\zeta_n^p$  over  $\mathbf{Q}$ , so  $g(T) \neq f(T)$ . The polynomials  $f(T)$  and  $g(T)$  are in  $\mathbf{Z}[T]$  since they both divide  $T^n - 1$  and any monic factor of  $T^n - 1$  in  $\mathbf{Q}[T]$  is in  $\mathbf{Z}[T]$  by Gauss' lemma.

Any  $n$ th root of unity is a root of  $T^n - 1$ , so  $f(T)$  and  $g(T)$  each divide  $T^n - 1$  in  $\mathbf{Q}[T]$ . They are different monic irreducibles, so  $T^n - 1 = f(T)g(T)h(T)$  for a monic  $h(T) \in \mathbf{Q}[T]$ . By Gauss' lemma,  $h(T) \in \mathbf{Z}[T]$ . Reducing this equation modulo  $p$ ,

$$(2.1) \quad T^n - \bar{1} = \bar{f}(T)\bar{g}(T)\bar{h}(T)$$

in  $\mathbf{F}_p[T]$ . The polynomial  $T^n - \bar{1}$  is separable in  $\mathbf{F}_p[T]$  since  $p$  doesn't divide  $n$ , so (2.1) tells us that  $\bar{f}(T)$  and  $\bar{g}(T)$  are relatively prime in  $\mathbf{F}_p[T]$ . Because  $f(T)$  and  $g(T)$  are monic, their reductions  $\bar{f}(T)$  and  $\bar{g}(T)$  have the same degrees as  $f(T)$  and  $g(T)$ , so the reductions are nonconstant.

Since  $g(\zeta_n^p) = 0$ ,  $g(T^p)$  has  $\zeta_n$  as a root, so  $f(T) | g(T^p)$  in  $\mathbf{Q}[T]$ . Write  $g(T^p) = f(T)k(T)$  for some monic  $k(T)$  in  $\mathbf{Q}[T]$ . We have  $k(T) \in \mathbf{Z}[T]$  by Gauss' lemma (why?). Reduce the equation  $g(T^p) = f(T)k(T)$  modulo  $p$  and use the formula  $\bar{g}(T^p) = \bar{g}(T)^p$  in  $\mathbf{F}_p[T]$  to get

$$\bar{g}(T)^p = \bar{f}(T)\bar{k}(T)$$

in  $\mathbf{F}_p[T]$ . Thus any irreducible factor of  $\bar{f}(T)$  in  $\mathbf{F}_p[T]$  is a factor of  $\bar{g}(T)$  (and there *are* irreducible factors since  $\bar{f}(T)$  is nonconstant). That contradicts relative primality of  $\bar{f}(T)$  and  $\bar{g}(T)$  in  $\mathbf{F}_p[T]$ .  $\square$

**Remark 2.6.** The proof of Theorem 2.5 above goes back to Dedekind [2]. Its appearance in van der Waerden's *Moderne Algebra* in 1930 made it the standard proof in later books. Here is another proof of Theorem 2.5, due to Landau [5]. Let  $f(T)$  be the minimal polynomial of  $\zeta_n$  over  $\mathbf{Q}$ , so  $f(T)$  is monic in  $\mathbf{Z}[T]$  as explained above. We want to show when  $(a, n) = 1$  that  $f(\zeta_n^a) = 0$ . For any integer  $k \geq 1$  write  $f(T^k) = f(T)Q_k(T) + R_k(T)$  in  $\mathbf{Z}[T]$ , where  $R_k = 0$  or  $\deg R_k < \deg f$ . Since  $R_k(T)$  is 0 or has degree less than  $\deg f = [\mathbf{Q}(\zeta_n) : \mathbf{Q}]$ ,  $R_k(T)$  is determined by  $R_k(\zeta_n) = f(\zeta_n^k)$ , so  $R_k(\zeta_n)$  only depends on  $k \bmod n$ . In particular, every  $R_k(T)$  is one of  $R_1(T), R_2(T), \dots, R_n(T)$ .

For any prime  $p$ ,  $R_p(\zeta_n) = f(\zeta_n^p) = f(\zeta_n^p) - f(\zeta_n)^p$ , so  $R_p(T)$  is the remainder when  $f(T^p) - f(T)^p$  is divided by  $f(T)$ . From  $f(T)^p \equiv f(T^p) \pmod{p}$  we get  $f(T^p) - f(T)^p \in p\mathbf{Z}[T]$ , which implies (why?) that  $R_p(T) \in p\mathbf{Z}[T]$ . Let  $C$  be the largest absolute value of any coefficient in  $R_1(T), R_2(T), \dots, R_n(T)$ . Since there are infinitely many primes (!), pick a prime  $p > C$ . The polynomial  $R_p(T)$  must be 0: its coefficients are smaller in absolute value than  $C$  and are divisible by  $p$ , which exceeds  $C$ . Therefore  $f(T^p) = f(T)Q_p(T)$  when  $p > C$ , so  $f(\zeta_n^p) = 0$ . This implies, by iteration, that  $f(\zeta_n^k) = 0$  for any positive integer  $k$  whose prime factors all exceed  $C$ . If  $(a, n) = 1$  and  $a > 1$ , define  $k := a + n \prod_{p \leq C, (p,a)=1} p$ . Then  $k \equiv a \pmod{n}$ , so  $(k, n) = 1$ . The two terms in the sum defining  $k$  are relatively prime, so every prime factor of this  $k$  is larger than  $C$  (why?), which implies  $0 = f(\zeta_n^k) = f(\zeta_n^a)$ .

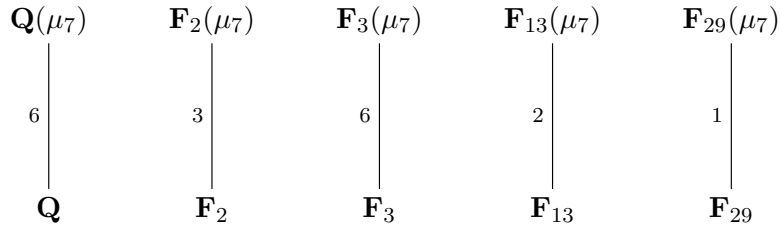
**Remark 2.7.** Any introductory discussion of cyclotomic extensions of  $\mathbf{Q}$  would not be complete without mentioning a deep theorem of Kronecker and Weber: every finite abelian extension of  $\mathbf{Q}$  lies inside a cyclotomic extension of  $\mathbf{Q}$ . This is false if the base field is any proper finite extension of  $\mathbf{Q}$ : when  $1 < [K : \mathbf{Q}] < \infty$  some finite abelian extension of  $K$  does not lie in a cyclotomic extension of  $K$ . For instance, if  $K = \mathbf{Q}(i)$  then  $K(\sqrt[4]{1+i})/K$  is abelian but for no  $n$  is  $K(\sqrt[4]{1+i}) \subset K(\zeta_n)$ .

Now we turn to Galois groups of cyclotomic extensions of the finite field  $\mathbf{F}_p$ .

**Theorem 2.8.** *When  $n$  is not divisible by the prime  $p$ , the image of  $\text{Gal}(\mathbf{F}_p(\mu_n)/\mathbf{F}_p)$  in  $(\mathbf{Z}/(n))^\times$  under the standard embedding is  $\langle p \pmod{n} \rangle$ . In particular,  $[\mathbf{F}_p(\mu_n) : \mathbf{F}_p]$  is the order of  $p \pmod{n}$ .*

*Proof.* The polynomial  $T^n - 1$  is separable in  $\mathbf{F}_p[T]$  and the general theory of finite fields tells us  $\text{Gal}(\mathbf{F}_p(\mu_n)/\mathbf{F}_p)$  is generated by the  $p$ th power map  $\varphi_p: x \mapsto x^p$  for all  $x$  in  $\mathbf{F}_p(\mu_n)$ . The standard embedding of  $\text{Gal}(\mathbf{F}_p(\mu_n)/\mathbf{F}_p)$  into  $(\mathbf{Z}/(n))^\times$  associates to  $\varphi_p$  the congruence class  $a \pmod{n}$  where  $\varphi_p(\zeta) = \zeta^a$  for all  $\zeta \in \mu_n$ . Then  $\zeta^p = \zeta^a$ , so  $a \equiv p \pmod{n}$ . Therefore the standard embedding of  $\text{Gal}(\mathbf{F}_p(\mu_n)/\mathbf{F}_p)$  into  $(\mathbf{Z}/(n))^\times$  turns  $\varphi_p$  into  $p \pmod{n}$ . Since  $\varphi_p$  generates the Galois group, the image of the Galois group in  $(\mathbf{Z}/(n))^\times$  is  $\langle p \pmod{n} \rangle$ , so the size of the Galois group is the order of  $p$  in  $(\mathbf{Z}/(n))^\times$ .  $\square$

**Example 2.9.** The degree  $[\mathbf{F}_p(\mu_7) : \mathbf{F}_p]$  is the order of  $p \pmod{7}$  that is 1, 2, 3, or 6 (if  $p \neq 7$ ). The field diagram below gives some examples.



These are all consistent with how  $(T^7 - 1)/(T - 1) = T^6 + T^5 + \dots + T + 1$  factors modulo each of the primes: into irreducibles of common degree  $[\mathbf{F}_p(\mu_7) : \mathbf{F}_p]$ .

$$\begin{aligned}
T^6 + T^5 + T^4 + T^3 + T^2 + T + 1 &\equiv (T^3 + T + 1)(T^3 + T^2 + 1) \pmod{2} \\
&\equiv \text{irreducible mod } 3 \\
&\equiv (T^2 + 3T + 1)(T^2 + 5T + 1)(T^2 + 6T + 1) \pmod{13} \\
&\equiv (T - 7)(T - 16)(T - 20)(T - 23)(T - 24)(T - 25) \pmod{29}.
\end{aligned}$$

For the cyclic group  $\text{Gal}(\mathbf{F}_p(\mu_n)/\mathbf{F}_p)$  to be as big as  $\text{Gal}(\mathbf{Q}(\mu_n)/\mathbf{Q}) \cong (\mathbf{Z}/(n))^\times$  is equivalent to saying  $\langle p \bmod n \rangle = (\mathbf{Z}/(n))^\times$ , so  $(\mathbf{Z}/(n))^\times$  must be a cyclic group and  $p \bmod n$  is a generator of it. The groups  $(\mathbf{Z}/(n))^\times$  are usually not cyclic (like  $n = 8$  or  $n = 15$ ), so the standard embedding  $\text{Gal}(\mathbf{F}_p(\mu_n)/\mathbf{F}_p) \hookrightarrow (\mathbf{Z}/(n))^\times$  is usually not surjective.

Theorem 2.8 generalizes to any finite field  $\mathbf{F}_q$  as a base field. A proof is left to the reader.

**Theorem 2.10.** *Let  $\mathbf{F}_q$  be a finite field with prime power order  $q$ . When  $n$  is relatively prime to  $q$ , the image of  $\text{Gal}(\mathbf{F}_q(\mu_n)/\mathbf{F}_q)$  in  $(\mathbf{Z}/(n))^\times$  is  $\langle q \bmod n \rangle$ . In particular,  $[\mathbf{F}_q(\mu_n) : \mathbf{F}_q]$  is the order of  $q \bmod n$ .*

### 3. COMPOSITES AND INTERSECTIONS OF CYCLOTOMIC EXTENSIONS

When  $T^n - 1$  and  $T^m - 1$  are separable over  $K$ , the composite field  $K(\mu_m)K(\mu_n)$  equals  $K(\mu_{[m,n]})$ . Indeed, both  $K(\mu_m)$  and  $K(\mu_n)$  lie in  $K(\mu_{[m,n]})$ , so their composite does too. For the reverse inclusion, a primitive root of unity of order  $[m, n]$  can be obtained by multiplying *suitable*  $m$ th and  $n$ th roots of unity (why?), so  $\mu_{[m,n]} \subset K(\mu_m)K(\mu_n)$ , which implies  $K(\mu_{[m,n]}) \subset K(\mu_m)K(\mu_n)$ . Therefore  $K(\mu_m)K(\mu_n) = K(\mu_{[m,n]})$ .

It is natural to guess that a counterpart of  $K(\mu_m)K(\mu_n) = K(\mu_{[m,n]})$  for intersections is  $K(\mu_m) \cap K(\mu_n) = K(\mu_{(m,n)})$ . The inclusion  $\supset$  is easy, but the other inclusion is not always true! It's possible for  $m$  and  $n$  to be relatively prime and  $K(\mu_m) \cap K(\mu_n)$  to be larger than  $K(\mu_1) = K$ . Matt Emerton pointed out to me the following simple example.

**Example 3.1.** If  $K = \mathbf{Q}(\sqrt{3})$  then  $K(\zeta_4) = K(i) = \mathbf{Q}(\sqrt{3}, i) = \mathbf{Q}(\sqrt{3}, \sqrt{-3}) = K(\zeta_3)$  because  $\zeta_3 = (-1 + \sqrt{-3})/2$ . Since  $K(\zeta_4)$  and  $K(\zeta_3)$  are equal and larger than  $K$  itself, their intersection is larger than  $K(\zeta_{(4,3)}) = K$ .

Here are some more examples.

**Example 3.2.** For  $n \geq 3$ , complex conjugation on  $\mathbf{Q}(\zeta_n)$  is an automorphism of order 2. The fixed field of complex conjugation on  $\mathbf{Q}(\zeta_n)$  is denoted  $\mathbf{Q}(\zeta_n)^+$ . Show as an exercise that  $\mathbf{Q}(\zeta_n)^+ = \mathbf{Q}(\zeta_n + \zeta_n^{-1})$  and  $[\mathbf{Q}(\zeta_n) : \mathbf{Q}(\zeta_n)^+] = 2$ . For each pair of relatively prime integers  $m$  and  $n$  that are both at least 3, a field  $K$  such that  $K(\mu_m) \cap K(\mu_n) \neq K$  is  $K = \mathbf{Q}(\zeta_{mn})^+$ . Since  $[\mathbf{Q}(\zeta_{mn}) : K] = 2$  and  $K \subset \mathbf{R}$ ,  $K(\zeta_m) = \mathbf{Q}(\zeta_{mn})$  and  $K(\zeta_n) = \mathbf{Q}(\zeta_{mn})$ . Thus  $K(\zeta_m) \cap K(\zeta_n) = \mathbf{Q}(\zeta_{mn})$  is larger than  $K$ . Taking  $m = 4$  and  $n = 3$  we get Example 3.1:  $\mathbf{Q}(\zeta_{12})^+ = \mathbf{Q}(\sqrt{3})$ .

**Example 3.3.** Using Theorem 2.8,

$$\mathbf{F}_3(\mu_5) \cap \mathbf{F}_3(\mu_7) = \mathbf{F}_{3^4} \cap \mathbf{F}_{3^6} = \mathbf{F}_{3^2} \neq \mathbf{F}_3.$$

This weirdness does not happen when the base field is the rational numbers.

**Theorem 3.4.** *For all positive integers  $m$  and  $n$ ,  $\mathbf{Q}(\mu_m) \cap \mathbf{Q}(\mu_n) = \mathbf{Q}(\mu_{(m,n)})$ ; in particular, if  $(m, n) = 1$  then  $\mathbf{Q}(\mu_m) \cap \mathbf{Q}(\mu_n) = \mathbf{Q}$ .*

*Proof.* By Theorem 2.5,  $[\mathbf{Q}(\mu_N) : \mathbf{Q}] = \#(\mathbf{Z}/(N))^\times = \varphi(N)$  for any positive integer  $N$ . There is a formula for  $\varphi(N)$  in terms of the prime factors of  $N$ :

$$(3.1) \quad \varphi(N) = N \prod_{p|N} \left(1 - \frac{1}{p}\right).$$

Since  $\mathbf{Q}(\mu_d) \subset \mathbf{Q}(\mu_m)$  when  $d|m$ , we have  $\mathbf{Q}(\mu_{(m,n)}) \subset \mathbf{Q}(\mu_m) \cap \mathbf{Q}(\mu_n)$ . To show this containment is an equality we will show  $\mathbf{Q}(\mu_m) \cap \mathbf{Q}(\mu_n)$  and  $\mathbf{Q}(\mu_{(m,n)})$  have the same degree over  $\mathbf{Q}$ .

For any finite Galois extensions  $L_1/K$  and  $L_2/K$  inside a common field,  $[L_1L_2 : K] = [L_1 : K][L_2 : K]/[L_1 \cap L_2 : K]$ . The composite field  $\mathbf{Q}(\mu_m)\mathbf{Q}(\mu_n)$  is  $\mathbf{Q}(\mu_{[m,n]})$ , so

$$[\mathbf{Q}(\mu_{[m,n]}) : \mathbf{Q}] = [\mathbf{Q}(\mu_m)\mathbf{Q}(\mu_n) : \mathbf{Q}] = \frac{[\mathbf{Q}(\mu_m) : \mathbf{Q}][\mathbf{Q}(\mu_n) : \mathbf{Q}]}{[\mathbf{Q}(\mu_m) \cap \mathbf{Q}(\mu_n) : \mathbf{Q}]}.$$

Replacing each  $[\mathbf{Q}(\mu_N) : \mathbf{Q}]$  on the right side with  $\varphi(N)$ ,

$$(3.2) \quad [\mathbf{Q}(\mu_m) \cap \mathbf{Q}(\mu_n) : \mathbf{Q}] = \frac{\varphi(m)\varphi(n)}{\varphi([m,n])}.$$

Using (3.1), (3.2) becomes

$$[\mathbf{Q}(\mu_m) \cap \mathbf{Q}(\mu_n) : \mathbf{Q}] = \frac{m \prod_{p|m} (1 - 1/p) \cdot n \prod_{p|n} (1 - 1/p)}{[m,n] \prod_{p|[m,n]} (1 - 1/p)}.$$

Since  $[m,n](m,n) = mn$ , the ratio  $mn/[m,n]$  is  $(m,n)$ . The prime factors of  $[m,n]$  are those dividing either  $m$  or  $n$ , so the ratio of products over primes is the product of  $1 - 1/p$  over all primes dividing both  $m$  and  $n$ , which means the prime factors of  $(m,n)$ . Therefore

$$[\mathbf{Q}(\mu_m) \cap \mathbf{Q}(\mu_n) : \mathbf{Q}] = (m,n) \prod_{p|(m,n)} \left(1 - \frac{1}{p}\right) = \varphi((m,n)),$$

which is  $[\mathbf{Q}(\mu_{(m,n)}) : \mathbf{Q}]$ , so  $\mathbf{Q}(\mu_m) \cap \mathbf{Q}(\mu_n)$  has the same degree over  $\mathbf{Q}$  as  $\mathbf{Q}(\mu_{(m,n)})$ , hence the fields are equal since we already saw one is a subfield of the other.  $\square$

#### 4. COINCIDENCES OF CYCLOTOMIC FIELDS OVER $\mathbf{Q}$

Knowing the degree of cyclotomic extensions of  $\mathbf{Q}$  lets us determine when two cyclotomic fields can coincide. For example,  $\mathbf{Q}(\zeta_3) = \mathbf{Q}(\zeta_6)$  since  $-\zeta_3$  has order 6. More simply,  $\mathbf{Q}(\zeta_1) = \mathbf{Q}(\zeta_2) = \mathbf{Q}$  since  $\zeta_1 = 1$  and  $\zeta_2 = -1$ . Here is the general result in this direction.

**Theorem 4.1.** *Let  $m$  and  $n$  be positive integers.*

- (1) *The number of roots of unity in  $\mathbf{Q}(\mu_m)$  is  $[2, m]$ .*
- (2) *If  $m \neq n$  then  $\mathbf{Q}(\mu_m) = \mathbf{Q}(\mu_n)$  if and only if  $\{m, n\} = \{k, 2k\}$  as sets for an odd  $k$ .*

*Proof.* 1) Our argument is adapted from [1, p. 158]. The root of unity  $-\zeta_m$  is in  $\mathbf{Q}(\mu_m)$  and it has order  $2m$  if  $m$  is odd, and  $m$  if  $m$  is even, hence the order of  $-\zeta_m$  is  $[2, m]$  in general. Therefore  $\mu_{[2,m]} \subset \mathbf{Q}(\mu_m)$ .

If  $\mathbf{Q}(\mu_m)$  contains an  $r$ th root of unity then  $\mathbf{Q}(\mu_r) \subset \mathbf{Q}(\mu_m)$ , and taking degrees over  $\mathbf{Q}$  shows  $\varphi(r) \leq \varphi(m)$ . As  $r \rightarrow \infty$ ,  $\varphi(r) \rightarrow \infty$  (albeit erratically<sup>2</sup>) so there is a largest  $r$  satisfying  $\mu_r \subset \mathbf{Q}(\mu_m)$ . Since  $\mu_m \mu_r = \mu_{[m,r]}$  is in  $\mathbf{Q}(\mu_m)$  we have  $[m,r] \leq r$ , so  $[m,r] = r$ . Thus  $r$  is a multiple of  $m$ . Write  $r = ms$ . By (3.1), for any  $a$  and  $b$  in  $\mathbf{Z}^+$  we have

$$\varphi(ab) = \frac{\varphi(a)\varphi(b)(a,b)}{\varphi((a,b))},$$

so

$$\varphi(r) = \varphi(ms) = \varphi(m)\varphi(s) \frac{(m,s)}{\varphi((m,s))} \geq \varphi(m)\varphi(s).$$

<sup>2</sup>This follows from showing any bound  $\varphi(r) \leq B$  also bounds  $r$  from above. For any prime power  $p^e$  dividing  $r$ ,  $\varphi(p^e) \leq B$  too since  $\varphi(p^e) | \varphi(r)$ , so  $p^{e-1}(p-1) \leq B$ . Then  $2^{e-1} \leq B$  and  $p-1 \leq B$ , so we get upper bounds on  $p$  and on  $e$ , which gives an upper bound on  $r$  by unique factorization.

Since  $\mathbf{Q}(\mu_m) = \mathbf{Q}(\mu_r)$  for the maximal  $r$ , computing degrees over  $\mathbf{Q}$  shows  $\varphi(m) = \varphi(r) \geq \varphi(m)\varphi(s)$ , so  $1 \geq \varphi(s)$ . Thus  $\varphi(s) = 1$ , so  $s = 1$  or  $2$ , so  $r = m$  or  $r = 2m$ . This shows the number of roots of unity in  $\mathbf{Q}(\mu_m)$  is either  $m$  or  $2m$ . If  $m$  is even then  $\varphi(2m) = 2\varphi(m) > \varphi(m)$ , so  $r \neq 2m$ . Thus when  $m$  is even the number of roots of unity in  $\mathbf{Q}(\mu_m)$  is  $m$ . If  $m$  is odd then  $-\zeta_m$  has order  $2m$ , so the number of roots of unity in  $\mathbf{Q}(\mu_m)$  is  $2m$ . In general the number of roots of unity in  $\mathbf{Q}(\mu_m)$  is  $[2, m]$ .

2) If  $\mathbf{Q}(\mu_m) = \mathbf{Q}(\mu_n)$  and  $m \neq n$  then counting roots of unity implies  $[2, m] = [2, n]$ . This becomes  $m = [2, n]$  for even  $m$  (so  $n = m/2$ ), and  $2m = [2, n]$  for odd  $m$  (so  $n = 2m$ ).  $\square$

**Remark 4.2.** Theorem 4.1 suggests two ways to parametrize cyclotomic extensions of  $\mathbf{Q}$  without duplication: as  $\mathbf{Q}(\mu_m)$  for  $m$  not twice an odd integer ( $m \not\equiv 2 \pmod{4}$ ) or for  $m$  equal to twice an odd integer ( $m \equiv 2 \pmod{4}$ ). In the first convention,  $\mathbf{Q}(\mu_m)$  contains  $2m$  roots of unity. The first convention, where  $m \not\equiv 2 \pmod{4}$ , is commonly used since certain important results about these fields take on a simpler appearance with that convention.

**Theorem 4.3.** *If  $E/\mathbf{Q}$  is a finite extension that contains no proper abelian extensions of  $\mathbf{Q}$  then  $\text{Gal}(E(\mu_n)/E) \cong (\mathbf{Z}/(n))^\times$  for all  $n \geq 1$ , or equivalently  $[E(\mu_n) : E] = \varphi(n)$ .*

*Proof.* From Galois theory, for finite extensions  $L/K$  and  $F/K$ ,  $[LF : F] = [L : L \cap F]$  if  $L/K$  is Galois. Therefore  $[E(\mu_n) : E] = [\mathbf{Q}(\mu_n)E : E] = [\mathbf{Q}(\mu_n) : \mathbf{Q}(\mu_n) \cap E]$ . The intersection  $\mathbf{Q}(\mu_n) \cap E$  is an abelian extension of  $\mathbf{Q}$  since every subfield of  $\mathbf{Q}(\mu_n)$  is abelian over  $\mathbf{Q}$ . Therefore by hypothesis  $\mathbf{Q}(\mu_n) \cap E = \mathbf{Q}$ , so  $[E(\mu_n) : E] = [\mathbf{Q}(\mu_n) : \mathbf{Q}] = \varphi(n)$ .  $\square$

**Example 4.4.** Any  $E/\mathbf{Q}$  of prime degree that is not Galois, such as  $\mathbf{Q}(\sqrt[p]{2})/\mathbf{Q}$  for prime  $p \geq 3$ , satisfies the hypothesis of Theorem 4.3 so  $\text{Gal}(E(\mu_n)/E) \cong (\mathbf{Z}/(n))^\times$  for  $n \geq 1$ .

## 5. CYCLOTOMIC POLYNOMIALS

In the complex numbers, all primitive  $n$ th roots of unity are  $\mathbf{Q}$ -conjugate and therefore have a common minimal polynomial in  $\mathbf{Q}[T]$ . It is called the  $n$ th *cyclotomic polynomial* and is denoted  $\Phi_n(T)$ . Explicitly,

$$\Phi_n(T) = \prod_{\substack{1 \leq k \leq n \\ (k, n) = 1}} (T - e^{2\pi i k/n}).$$

The first few cyclotomic polynomials are

$$\Phi_1(T) = T - 1, \quad \Phi_2(T) = T + 1, \quad \Phi_3(T) = T^2 + T + 1, \quad \Phi_4(T) = T^2 + 1.$$

For all  $n \geq 1$ ,  $\Phi_n(T)$  is monic of degree  $\varphi(n)$  in  $\mathbf{Q}[T]$  and  $\Phi_n(T)$  is irreducible over  $\mathbf{Q}$ . Here are some identities involving these polynomials, where  $p$  is a prime:

- (1)  $T^n - 1 = \prod_{d|n} \Phi_d(T)$ ,
- (2)  $\Phi_n(T) = T^{\varphi(n)} \Phi_n(1/T)$  for  $n \geq 2$ ,
- (3)  $\Phi_p(T) = T^{p-1} + T^{p-2} + \cdots + T + 1$ ,
- (4)  $\Phi_{p^r}(T) = \Phi_p(T^{p^{r-1}})$ ,
- (5)  $\Phi_{2n}(T) = \Phi_n(-T)$  for odd  $n$ ,
- (6)  $\Phi_{p_1^{r_1} \cdots p_k^{r_k}}(T) = \Phi_{p_1 \cdots p_k}(T^{p_1^{r_1-1} \cdots p_k^{r_k-1}})$ ,
- (7) if  $(p, m) = 1$  then  $\Phi_{p^r m}(T) = \Phi_m(T^{p^r})/\Phi_m(T^{p^{r-1}})$ ,

Except for the first formula, these identities can be checked by showing the right side has the correct degree and one correct root to be the cyclotomic polynomial on the left side.



(Two monic irreducible polynomials with a common root are equal.) The first identity can be regarded as a recursive definition of the cyclotomic polynomials.

**Example 5.1.** Since  $\Phi_2(T) = T + 1$ , we have  $\Phi_8(T) = \Phi_2(T^4) = T^4 + 1$ . Since  $\Phi_3(T) = T^2 + T + 1$ ,  $\Phi_6(T) = \Phi_3(-T) = T^2 - T + 1$  and  $\Phi_{24}(T) = \Phi_6(T^4) = \Phi_3(-T^4) = T^{12} - T^4 + 1$ .

Here are the first 10 cyclotomic polynomials.

$$\begin{array}{ll} \Phi_1(T) = T - 1 & \Phi_2(T) = T + 1 \\ \Phi_3(T) = T^2 + T + 1 & \Phi_4(T) = T^2 + 1 \\ \Phi_5(T) = T^4 + T^3 + T^2 + T + 1 & \Phi_6(T) = T^2 - T + 1 \\ \Phi_7(T) = T^6 + T^5 + T^4 + T^3 + T^2 + T + 1 & \Phi_8(T) = T^4 + 1 \\ \Phi_9(T) = T^6 + T^3 + 1 & \Phi_{10}(T) = T^4 - T^3 + T^2 - T + 1 \end{array}$$

**Theorem 5.2.** For all  $n \geq 1$ ,  $\Phi_n(T) \in \mathbf{Z}[T]$ .

*Proof.* Let  $\zeta_n$  be a primitive  $n$ th root of unity in  $\mathbf{C}$ , so  $\Phi_n(T)$  is its minimal polynomial over  $\mathbf{Q}$ . Since  $T^n - 1$  vanishes at  $\zeta_n$ ,  $\Phi_n(T)$  divides  $T^n - 1$  in  $\mathbf{Q}[T]$ . Since  $\Phi_n(T)$  and  $T^n - 1$  are both monic and  $T^n - 1 \in \mathbf{Z}[T]$ , by Gauss' lemma  $\Phi_n(T) \in \mathbf{Z}[T]$ .  $\square$

The sequence of cyclotomic polynomials provide an interesting example where initial data can be misleading. The coefficients of the first 100 cyclotomic polynomials are 0 or  $\pm 1$ , but this pattern is not true in general! For instance,  $\Phi_{105}(T)$  has a coefficient  $-2$  for  $T^{41}$  and  $T^7$  (the other coefficients are 0 and  $\pm 1$ ). Why does it take so long for a coefficient besides 0 and  $\pm 1$  to occur? Well, the fourth and fifth cyclotomic polynomial identities above show the nonzero coefficients of all cyclotomic polynomials are determined by the coefficients of the  $\Phi_n(T)$  where  $n$  is a product of distinct odd primes. The polynomial  $\Phi_p(T)$  only has coefficient 1 and it can be shown [4] that  $\Phi_{pq}(T)$  only has coefficients 0 and  $\pm 1$  where  $p$  and  $q$  are different primes. Therefore any  $n$  with at most 2 odd prime factors only has coefficients among 0 and  $\pm 1$ . The first positive integer that does not have at most 2 odd prime factors is  $3 \cdot 5 \cdot 7 = 105 > 100$ , which shows  $\Phi_{105}(T)$  is the first cyclotomic polynomial that even has a chance to have a coefficient other than 0 and  $\pm 1$ . By a theorem of Schur, if  $n$  has  $t$  odd prime factors then  $\Phi_n(T)$  has coefficient  $-(t-1)$  (thus predicting the coefficient of  $-2$  in  $\Phi_{105}(T)$ ). To produce large coefficients in  $\Phi_n(T)$  we should give  $n$  a lot of odd prime factors and numbers below 100 have at most 2 odd prime factors.

Cyclotomic polynomials for prime-power  $n$ , say  $n = p^r$ , can be written down concretely:

$$\Phi_{p^r}(T) = \frac{T^{p^r} - 1}{T^{p^{r-1}} - 1} = \sum_{k=0}^{p-1} T^{p^{r-1}k}.$$

**Theorem 5.3.** The polynomial  $\Phi_{p^r}(T + 1)$  is Eisenstein with respect to  $p$ .

*Proof.* The constant term of  $\Phi_{p^r}(T + 1)$  is

$$\Phi_{p^r}(1) = \sum_{k=0}^{p-1} 1^{p^{r-1}k} = p,$$

which is divisible by  $p$  just once. To show the non-leading coefficients are all multiples of  $p$ , we reduce the coefficients mod  $p$ . Since, in  $\mathbf{F}_p[T]$ ,  $T^{p^r} - 1 = (T - 1)^{p^r}$  and  $T^{p^{r-1}} - 1 =$

$(T - 1)^{p^{r-1}}$ , we have (reducing coefficients mod  $p$ )

$$\overline{\Phi}_{p^r}(T) = \frac{T^{p^r} - 1}{T^{p^{r-1}} - 1} = (T - 1)^{p^r - p^{r-1}} \text{ in } \mathbf{F}_p[T],$$

so

$$\overline{\Phi}_{p^r}(T + 1) = T^{p^r - p^{r-1}} \text{ in } \mathbf{F}_p[T].$$

The degree of  $\overline{\Phi}_{p^r}(T + 1)$  is  $p^r - p^{r-1}$ , so all its non-leading coefficients are 0 in  $\mathbf{F}_p$ , which means the coefficients as integers are multiples of  $p$ .  $\square$

Using the Eisenstein irreducibility criterion,  $\overline{\Phi}_{p^r}(T + 1)$  is irreducible in  $\mathbf{Q}[T]$ , so  $\overline{\Phi}_{p^r}(T)$  is irreducible in  $\mathbf{Q}[T]$ . Therefore  $[\mathbf{Q}(\mu_{p^r}) : \mathbf{Q}] = p^r - p^{r-1} = \#(\mathbf{Z}/(p^r))^\times$ , so the embedding  $\text{Gal}(\mathbf{Q}(\mu_{p^r})/\mathbf{Q}) \hookrightarrow (\mathbf{Z}/(p^r))^\times$  is an isomorphism. This is an alternate proof of Theorem 2.5 when  $n$  is a prime power that is simpler than the proof we gave before.

Cyclotomic polynomials can be used to prove some results that don't appear to be about roots of unity in the first place. One such result is an elementary proof that for any  $n > 1$  there are infinitely many primes  $p \equiv 1 \pmod n$  [6, Cor. 2.11]. A second result is a proof of Wedderburn's theorem that all finite division rings are commutative [3, Thm. 13.1].

Since cyclotomic polynomials are in  $\mathbf{Z}[T]$ , let's reduce them modulo  $p$  and ask how they factor. It suffices to look at  $\overline{\Phi}_n(T) = \Phi_n(T) \pmod p$  when  $(p, n) = 1$  since reducing the seventh algebraic identity for cyclotomic polynomials at the start of this section gives us

$$(5.1) \quad \overline{\Phi}_{p^r m}(T) = \overline{\Phi}_m(T)^{p^r - p^{r-1}} \pmod p$$

in  $\mathbf{F}_p[T]$  when  $(p, m) = 1$ .

**Theorem 5.4.** *When the prime  $p$  does not divide  $n$ , the monic irreducible factors of  $\overline{\Phi}_n(T) \in \mathbf{F}_p[T]$  are distinct and each has degree equal to the order of  $p \pmod n$ .*

*Proof.* Since  $\overline{\Phi}_n(T) \mid (T^n - 1)$  in  $\mathbf{Z}[T]$ , this divisibility relation is preserved when reducing modulo  $p$ , so  $\overline{\Phi}_n(T)$  is separable in  $\mathbf{F}_p[T]$  because  $T^n - \overline{1}$  is separable in  $\mathbf{F}_p[T]$ . (Here we need  $(p, n) = 1$ .)

Let  $\alpha$  be a root of  $\overline{\Phi}_n(T)$  in an extension of  $\mathbf{F}_p$ . We will show that  $\alpha$  inherits the expected algebraic property of being a primitive  $n$ th root of unity. Since  $\overline{\Phi}_n(T) \mid T^n - \overline{1}$ , from  $\overline{\Phi}_n(\alpha) = 0$  we have  $\alpha^n = 1$ . If  $\alpha$  were not of order  $n$  then it has some order  $m$  that properly divides  $n$ . Then  $\alpha$  is a root of  $T^m - \overline{1} = \prod_{d \mid m} \overline{\Phi}_d(T)$ , so  $\overline{\Phi}_d(\alpha) = 0$  for some  $d$  properly dividing  $n$ . Since  $d \mid n$ ,  $T^n - \overline{1}$  is divisible by  $\overline{\Phi}_n(T)\overline{\Phi}_d(T)$ , so  $\alpha$  is a double root of  $T^n - \overline{1}$ , but  $T^n - \overline{1}$  has no repeated roots. Therefore we have a contradiction, so  $\alpha$  is a primitive  $n$ th root of unity.

Let  $\pi(T)$  be an irreducible factor of  $\overline{\Phi}_n(T)$  in  $\mathbf{F}_p[T]$  and let  $\alpha$  denote a root of  $\pi(T)$ . Then  $\alpha$  is a primitive  $n$ th root of unity, so  $\deg \pi = [\mathbf{F}_p(\alpha) : \mathbf{F}_p]$  is the order of  $p \pmod n$  by Theorem 2.8.  $\square$

**Example 5.5.** The polynomial  $\Phi_5(T) = T^4 + T^3 + T^2 + T + 1$  factors over  $\mathbf{F}_p$  into irreducible factors whose degrees equal the order of  $p \pmod 5$ . For example,  $T^4 + T^3 + T^2 + T + 1$  is irreducible in  $\mathbf{F}_3[T]$  since  $3 \pmod 5$  has order 4, while

$$T^4 + T^3 + T^2 + T + 1 = (T - 3)(T - 4)(T - 5)(T - 9)$$

in  $\mathbf{F}_{11}[T]$  with irreducible factors of degree 1 since  $11 \pmod 5$  has order 1, and

$$T^4 + T^3 + T^2 + T + 1 = (T^2 + 5T + 1)(T^2 + 15T + 1)$$

in  $\mathbf{F}_{19}[T]$  with irreducible factors of degree 2 since  $19 \pmod 5$  has order 2.

**Example 5.6.** The polynomial  $\Phi_7(T) = T^6 + T^5 + T^4 + T^3 + T^2 + T + 1$  factors over  $\mathbf{F}_p$  into irreducible factors whose degrees equal the order of  $p \bmod 7$ . For example,  $2 \bmod 7$  has order 3 so  $\Phi_7(T)$  factors over  $\mathbf{F}_2$  into a product of irreducible cubics:

$$T^6 + T^5 + T^4 + T^3 + T^2 + T + 1 = (T^3 + T + 1)(T^3 + T^2 + 1)$$

in  $\mathbf{F}_2[T]$ . This explains what happened in Example 2.9: if  $\zeta$  is a primitive 7th root of unity in characteristic 2, then it and  $\zeta^3$  are roots of the two different cubics on the right side: one has roots  $\zeta, \zeta^2$ , and  $\zeta^4$ , while the other has roots  $\zeta^3, (\zeta^3)^2 = \zeta^6$ , and  $(\zeta^3)^4 = \zeta^5$ .

**Corollary 5.7.** *The reduction  $\overline{\Phi}_n(T)$  is irreducible in  $\mathbf{F}_p[T]$  if and only if  $(p, n) = 1$  and  $p \bmod n$  is a generator of  $(\mathbf{Z}/(n))^\times$ .*

*Proof.* If  $\overline{\Phi}_n(T)$  is irreducible in  $\mathbf{F}_p[T]$  then  $(p, n) = 1$  by (5.1), so Theorem 5.4 tells us the order of  $p \bmod n$  is  $\varphi(n)$ :  $p \bmod n$  generates  $(\mathbf{Z}/(n))^\times$ . Conversely, if  $(p, n) = 1$  and  $p \bmod n$  is a generator of  $(\mathbf{Z}/(n))^\times$  then Theorem 5.4 tells us the irreducible factors of  $\overline{\Phi}_n(T)$  in  $\mathbf{F}_p[T]$  have degree  $\varphi(n) = \deg(\overline{\Phi}_n(T))$ , so  $\overline{\Phi}_n(T)$  is irreducible.  $\square$

Thus many cyclotomic polynomials are examples of irreducible polynomials in  $\mathbf{Z}[T]$  that factor modulo *every* prime: if  $(\mathbf{Z}/(n))^\times$  is not a cyclic group then there is no generator for  $(\mathbf{Z}/(n))^\times$ , so Corollary 5.7 says there is no prime  $p$  such that  $\Phi_n(T) \bmod p$  is irreducible. In other words,  $\Phi_n(T) \bmod p$  factors for all primes  $p$ .

**Example 5.8.** The least  $n$  such that  $(\mathbf{Z}/(n))^\times$  is non-cyclic is  $n = 8$ , and  $\Phi_8(T) = T^4 + 1$ . This polynomial is reducible mod  $p$  for all  $p$ . Here are some factorizations:

$$\begin{aligned} \Phi_8(T) &\equiv (T + 1)^4 \pmod{2}, \\ \Phi_8(T) &\equiv (T^2 + T + 2)(T^2 + 2T + 2) \pmod{3}, \\ \Phi_8(T) &\equiv (T^2 + 2)(T^2 + 3) \pmod{5}, \\ \Phi_8(T) &\equiv (T^2 + 3T + 1)(T^2 + 4T + 1) \pmod{7}, \\ \Phi_8(T) &\equiv (T^2 + 3T + 10)(T^2 + 8T + 10) \pmod{11}, \\ \Phi_8(T) &\equiv (T - 2)(T - 8)(T - 9)(T - 15) \pmod{17}, \\ \Phi_8(T) &\equiv (T^2 + 6T + 18)(T^2 + 13T + 18) \pmod{19}, \\ \Phi_8(T) &\equiv (T^2 + 5T + 1)(T^2 + 18T + 1) \pmod{23}, \\ \Phi_8(T) &\equiv (T^2 + 12)(T^2 + 17) \pmod{29}, \\ \Phi_8(T) &\equiv (T^2 + 8T + 1)(T^2 + 23T + 1) \pmod{31}. \end{aligned}$$

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