ACCELERATING CONVERGENCE OF SERIES

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1. Introduction

An infinite series is the limit of its partial sums. However, it may take a large number of terms to get even a few correct digits for the series from its partial sums. For example,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \tag{1.1}$$

converges but the partial sums $s_N = 1 + 1/4 + 1/9 + \cdots + 1/N^2$ take a long time to settle down, as the following table illustrates, where we truncate s_N to 8 digits after the decimal point. The partial sum s_{1000} turns out to agree with the full series (1.1) only in 1.64.

N	10	20	25	50	100	1000
s_N	1.54976773	1.59616324	1.60572340	1.62513273	1.63498390	1.64393456

That the partial sums s_N converge slowly is related to the error bound in the integral test: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = s_N + r_N$ where

$$r_N = \sum_{n=N+1}^{\infty} \frac{1}{n^2} < \int_N^{\infty} \frac{dx}{x^2} = \frac{1}{N}. \tag{1.2}$$

To approximate (1.1) correctly to 3 digits after the decimal point we want $r_N < .0001 = 1/10^4$, so the bound in (1.2) compels us to make $1/N \leq 1/10^4$, so $N \geq 10000$.[1] In the era before electronic computers, computing the 10000th partial sum of (1.1) was not feasible.

Our theme will be ways to speed up the convergence of a series by replacing its terms with new terms whose partial sums converge more rapidly. Such techniques are called series acceleration methods. We will accelerate (1.1), for example, so a 10th accelerated partial sum is more accurate than s_{1000}.

2. Series with positive terms: Kummer's transformation

Let $\displaystyle\sum_{n=1}^{\infty} a_n$ be a convergent series whose terms a_n are positive. If $\{b_n\}$ is a sequence growing at the same rate as $\{a_n\}$, meaning $\dfrac{a_n}{b_n} \to 1$ as $n \to \infty$, then $\displaystyle\sum_{n=1}^{\infty} b_n$ converges by the limit comparison test. If we happen to know the exact value of $B = \displaystyle\sum_{n=1}^{\infty} b_n$, then

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n + \sum_{n=1}^{\infty} (a_n - b_n) = B + \sum_{n=1}^{\infty} \left(1 - \frac{b_n}{a_n}\right) a_n \tag{2.1}$$

[1] The lower bound $r_N > \int_{N+1}^{\infty} dx/x^2 = 1/(N+1)$ proves $r_N < .0001 \Rightarrow N + 1 > 10000$, so $N \geq 10000$.

and the series on the right is likely to converge more rapidly than the series on the left since its terms tend to 0 more quickly than a_n on account of the new factor $1-b_n/a_n$, which tends to 0. The identity (2.1) goes back to Kummer [5] and is called Kummer's transformation.

Example 2.1. We will use (2.1) to rewrite (1.1) as a new series where the remainder for the Nth partial sum decays faster than $1/N$.

A series whose terms grow at the same rate as (1.1) is $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$, which has exact value $B = 1$ from the simplest example of a telescoping series:

$$\sum_{n=1}^{N} \frac{1}{n(n+1)} = \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{N} \to 1 \tag{2.2}$$

as $N \to \infty$. Taking $a_n = \frac{1}{n^2}$ and $b_n = \frac{1}{n(n+1)}$, so $\frac{b_n}{a_n} = \frac{n}{n+1}$, (2.1) says

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} + \sum_{n=1}^{\infty} \left(1 - \frac{n}{n+1}\right)\frac{1}{n^2} = 1 + \sum_{n=1}^{\infty} \frac{1}{n^2(n+1)}. \tag{2.3}$$

Letting $s'_N = 1 + \sum_{n=1}^{N} \frac{1}{n^2(n+1)}$, here are its values (truncated to 8 digits after the decimal point) for the same N as in the previous table. This seems to converge faster than s_N.

N	10	20	25	50	100	1000
s'_N	1.64067682	1.64378229	1.64418494	1.64474057	1.64488489	1.64493356

The bound on the remainder $r'_N = \sum_{n=N+1}^{\infty} \frac{1}{n^2(n+1)}$ is smaller than $1/N$:

$$r'_N < \sum_{n=N+1}^{\infty} \frac{1}{n^3} < \int_N^{\infty} \frac{dx}{x^3} = \frac{1}{2N^2}. \tag{2.4}$$

Therefore $r'_N < .0001$ if $1/(2N^2) \leq .0001$, which is equivalent to $N \geq 71$, and that's a great improvement on the bound $N \geq 10000$ to make $r_N < .0001$. Since $s'_{71} = 1.644837\ldots$, the series (1.1) lies between $1.644737\ldots$ and $1.644937\ldots$, and since $1/(2N^2) = .0000005$ when $N = 1000$, the value of s'_{1000} tells us (1.1) is 1.64493 to 5 digits after the decimal point. By accelerating the series further we'll find that same approximation using a much earlier partial sum than the 1000-th.

From the series on the right in (2.3), let $a_n = \frac{1}{n^2(n+1)}$. A sequence growing at the same rate is $b_n = \frac{1}{n(n+1)(n+2)}$, and we can find $\sum_{n=1}^{\infty} b_n$ exactly by a telescoping series: as $N \to \infty$,

$$\sum_{n=1}^{N} b_n = \sum_{n=1}^{N} \left(\frac{1/2}{n(n+1)} - \frac{1/2}{(n+1)(n+2)}\right) = \frac{1}{4} - \frac{1/2}{(N+1)(N+2)} \to \frac{1}{4}. \tag{2.5}$$

In (2.1) with $a_n = \frac{1}{n^2(n+1)}$ and $b_n = \frac{1}{n(n+1)(n+2)}$, we have $B = \frac{1}{4}$ and $\frac{b_n}{a_n} = \frac{n}{n+2}$:

$$\sum_{n=1}^{\infty} \frac{1}{n^2(n+1)} = \frac{1}{4} + \sum_{n=1}^{\infty} \left(1 - \frac{n}{n+2}\right)\frac{1}{n^2(n+1)} = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{2}{n^2(n+1)(n+2)}.$$

Feeding this into the right side of (2.3),

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \sum_{n=1}^{\infty} \frac{2}{n^2(n+1)(n+2)}. \tag{2.6}$$

When $s''_N = 1 + \frac{1}{4} + \sum_{n=1}^{N} \frac{2}{n^2(n+1)(n+2)}$, the next table exhibits faster convergence than previous tables for s_N and s''_N.

N	10	20	25	50	100	1000
s''_N	1.64446470	1.64486454	1.64489719	1.64492911	1.64493342	1.64493406

Letting $r''_N = \sum_{n=N+1}^{\infty} \frac{2}{n^2(n+1)(n+2)}$, we have

$$r''_N < \sum_{n=N+1}^{\infty} \frac{2}{n^4} < \int_N^{\infty} \frac{2}{x^4}\, dx = \frac{2}{3N^3},$$

which improves on (2.4) by an extra power of N just as (2.4) improved on (1.2) by an extra power of N. We have $r''_N < .0001$ if $2/(3N^3) < .0001$, which is equivalent to $N \geq 19$, so from the value of s''_{20} in the table above, (1.1) is between $s''_{20} - .0001 = 1.64476\ldots$ and $s''_{20} + .0001 = 1.644964\ldots$.

Let's accelerate the series on the right in (2.6): for $a_n = \frac{2}{n^2(n+1)(n+2)}$, a sequence growing at the same rate that is exactly summable is $b_n = \frac{2}{n(n+1)(n+2)(n+3)}$, where

$$B = \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \left(\frac{2/3}{n(n+1)(n+2)} - \frac{2/3}{(n+1)(n+2)(n+3)} \right) = \frac{1}{9}$$

and $\frac{b_n}{a_n} = \frac{n}{n+3}$, so (2.1) tells us

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{2}{n^2(n+1)(n+2)} &= \frac{1}{9} + \sum_{n=1}^{\infty} \left(1 - \frac{n}{n+3}\right) \frac{2}{n^2(n+1)(n+2)} \\ &= \frac{1}{9} + \sum_{n=1}^{\infty} \frac{6}{n^2(n+1)(n+2)(n+3)}. \end{aligned}$$

Feeding this into (2.6),

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \sum_{n=1}^{\infty} \frac{6}{n^2(n+1)(n+2)(n+3)}. \tag{2.7}$$

Setting $s'''_N = 1 + \frac{1}{4} + \frac{1}{9} + \sum_{n=1}^{N} \frac{6}{n^2(n+1)(n+2)(n+3)}$, we have the following values.

N	10	20	25	50	100	1000
s'''_N	1.64485320	1.64492728	1.64493110	1.64493385	1.64493405	1.64493406

The remainder $r_N''' = \displaystyle\sum_{n=N+1}^{\infty} \frac{6}{n^2(n+1)(n+2)(n+3)}$ has the bound

$$r_N''' < \sum_{n=N+1}^{\infty} \frac{6}{n^5} < \int_N^{\infty} \frac{6}{x^5}\,dx = \frac{6}{4N^4} = \frac{3}{2N^4},$$

so $r_{25}''' < .00000384$. From the value of s_{25}''' in the table, (1.1) is between 1.64492726 and 1.64493495, which tells us (1.1) is 1.6449 to 4 digits after the decimal point.

We can continue this process. For each $k \geq 1$, telescoping series like (2.2) and (2.5) generalize to

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n(n+1)\cdots(n+k)} &= \sum_{n=1}^{\infty}\left(\frac{1/k}{n(n+1)\cdots(n+k-1)} - \frac{1/k}{(n+1)(n+2)\cdots(n+k)}\right) \\ &= \frac{1}{k\cdot k!} \end{aligned} \tag{2.8}$$

and this lets us generalize (2.3), (2.6), and (2.7) to

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{j=1}^{k} \frac{1}{j^2} + \sum_{n=1}^{\infty} \frac{k!}{n^2(n+1)(n+2)\cdots(n+k)} \tag{2.9}$$

for each $k \geq 0$, where the first sum on the left is 0 at $k = 0$. The remainder term $r_N^{(k)}$ for the Nth partial sum of the rightmost series in (2.9) satisfies

$$r_N^{(k)} < \int_N^{\infty} \frac{k!}{x^{k+2}}\,dx = \frac{k!/(k+1)}{N^{k+1}}. \tag{2.10}$$

Put $k = 5$ in (2.9) and let $s_N^{(5)} = \displaystyle\sum_{n=1}^{5} \frac{1}{n^2} + \sum_{n=1}^{N} \frac{120}{n^2(n+1)(n+2)(n+3)(n+4)(n+5)}$. We get the following values.

N	10	20	25	50	100	1000
$s_N^{(5)}$	1.64492895	1.64493391	1.64493402	1.64493406	1.64493406	1.64493406

By (2.10), the error for the 20th partial sum is at most $(120/6)/20^6 = 1/20^5 = .0000003125$, so (1.1) is between $1.6449335\ldots$ and $1.6449342\ldots$.

The series (1.1) that we have been finding good approximations to has an exact formula: $\dfrac{\pi^2}{6} = 1.6449340\ldots$. This beautiful and unexpected result was discovered by a young Euler in 1735, when he was still in his 20s, and it is what first made him famous. Before determining the exact value, Euler created an acceleration method in 1731 to estimate the series to 6 digits after the decimal point, which was far beyond direct hand calculation of (1.1). (Figure 1 shows this estimate on the second line, taken from the end of his article.) An account of this work is in [6], and the original paper can be read (in English translation) at [1].

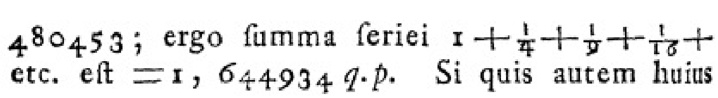
480453; ergo ſumma ſeriei $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} +$ etc. eſt $= 1,644934$ q.p. Si quis autem huius

FIGURE 1. End of Euler's article where $\sum_{n\geq 1} 1/n^2$ is estimated as 1.644934.

Example 2.2. Consider $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^3}$. Unlike (1.1), there is no known simple formula for this series in terms of more familiar numbers. We will accelerate the series four times.

The nth term $a_n = \dfrac{1}{n^3}$ grows at the same rate as $b_n = \dfrac{1}{n(n+1)(n+2)}$, and we know the exact value of $\displaystyle\sum_{n=1}^{\infty} b_n$: by (2.5), it is $\dfrac{1}{4}$, so by (2.1)

$$\sum_{n=1}^{\infty}\frac{1}{n^3} = \frac{1}{4} + \sum_{n=1}^{\infty}\left(1-\frac{b_n}{a_n}\right)a_n = \frac{1}{4} + \sum_{n=1}^{\infty}\frac{3n+2}{n^3(n+1)(n+2)}.$$

Now let $a_n = \dfrac{3n+2}{n^3(n+1)(n+2)}$, so a_n grows like $\dfrac{3}{n^4}$. A sequence growing at the same rate whose sum we know is $b_n = \dfrac{3}{n(n+1)(n+2)(n+3)}$: by (2.8),

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\frac{3}{n(n+1)(n+2)(n+3)} = \frac{3}{3\cdot 3!} = \frac{1}{6},$$

so by (2.1) and algebra

$$\sum_{n=1}^{\infty}\frac{1}{n^3} = \frac{1}{4} + \frac{1}{6} + \sum_{n=1}^{\infty}\left(1-\frac{b_n}{a_n}\right)a_n = \frac{1}{4} + \frac{1}{6} + \sum_{n=1}^{\infty}\frac{11n+6}{n^3(n+1)(n+2)(n+3)}.$$

Next let $a_n = \dfrac{11n+6}{n^3(n+1)(n+2)(n+3)}$, which grows like $\dfrac{11}{n^5}$. A sequence growing at the same rate whose sum we know is $b_n = \dfrac{11}{n(n+1)(n+2)(n+3)(n+4)}$: by (2.8),

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\frac{11}{n(n+1)(n+2)(n+3)(n+4)} = \frac{11}{4\cdot 4!} = \frac{11}{96},$$

so by (2.1) and algebra

$$\sum_{n=1}^{\infty}\frac{1}{n^3} = \frac{1}{4} + \frac{1}{6} + \frac{11}{96} + \sum_{n=1}^{\infty}\left(1-\frac{b_n}{a_n}\right)a_n = \frac{1}{4} + \frac{1}{6} + \frac{11}{96} + \sum_{n=1}^{\infty}\frac{50n+24}{n^3(n+1)\cdots(n+4)}.$$

It is left to the reader to derive the next acceleration, which is

$$\sum_{n=1}^{\infty}\frac{1}{n^3} = \frac{1}{4} + \frac{1}{6} + \frac{11}{96} + \frac{1}{12} + \sum_{n=1}^{\infty}\frac{274n+120}{n^3(n+1)\cdots(n+5)}.$$

We now have five partial sums that each tend to $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^3}$:

$$s_N = \sum_{n=1}^{N}\frac{1}{n^3}, \quad s_N' = \frac{1}{4} + \sum_{n=1}^{N}\frac{3n+2}{n^3(n+1)(n+2)}, \quad s_N'' = \frac{1}{4} + \frac{1}{6} + \sum_{n=1}^{N}\frac{11n+6}{n^3(n+1)(n+2)(n+3)},$$

$$\begin{aligned} s_N''' &= \frac{1}{4} + \frac{1}{6} + \frac{11}{96} + \sum_{n=1}^{N}\frac{50n+24}{n^3(n+1)(n+2)(n+3)(n+4)}, \\ s_N^{(4)} &= \frac{1}{4} + \frac{1}{6} + \frac{11}{96} + \frac{1}{12} + \sum_{n=1}^{N}\frac{274n+120}{n^3(n+1)(n+2)(n+3)(n+4)(n+5)}. \end{aligned}$$

The table below compares these partial sums for several values of N, each partial sum being truncated (not rounded) to 8 digits after the decimal point.

N	10	20	25	50	100	1000
s_N	1.19753198	1.20086784	1.20128826	1.20186086	1.20200740	1.20205640
s'_N	1.20131986	1.20195009	1.20200051	1.20204940	1.20205593	1.20205690
s''_N	1.20190261	1.20204420	1.20205138	1.20205651	1.20205687	1.20205690
s'''_N	1.20201708	1.20205498	1.20205621	1.20205687	1.20205690	1.20205690
$s^{(4)}_N$	1.20204483	1.20205655	1.20205679	1.20205690	1.20205690	1.20205690

We can bound the remainder term for each partial sum using the integral test, as in our previous example:

$$r_N := \sum_{n=N+1}^{\infty} \frac{1}{n^3} < \int_N^{\infty} \frac{dx}{x^3} = \frac{1}{2N^2},$$

$$r'_N := \sum_{n=N+1}^{\infty} \frac{3n+2}{n^3(n+1)(n+2)} < \sum_{n=N+1}^{\infty} \frac{3}{n^3(n+2)} < \int_N \frac{3}{x^4}\,dx = \frac{1}{N^3},$$

$$r''_N := \sum_{n=N+1}^{\infty} \frac{11n+6}{n^3(n+1)(n+2)(n+3)} < \sum_{n=N+1}^{\infty} \frac{11}{n^3(n+2)(n+3)} < \int_N \frac{11}{x^5}\,dx = \frac{11}{4N^4},$$

$$r'''_N := \sum_{n=N+1}^{\infty} \frac{50n+24}{n^3(n+1)(n+2)(n+3)(n+4)} < \sum_{n=N+1}^{\infty} \frac{50}{n^6} < \int_N \frac{50}{x^6}\,dx = \frac{50}{5N^5} = \frac{10}{N^5},$$

and

$$r^{(4)}_N := \sum_{n=N+1}^{\infty} \frac{274n+120}{n^3(n+1)(n+2)\cdots(n+5)} < \sum_{n=N+1}^{\infty} \frac{274}{n^7} < \int_N \frac{274}{x^7}\,dx = \frac{274}{6N^6} = \frac{137}{3N^6}.$$

For example, the bounds imply $r_N < .00001$ for $N \geq 224$, $r'_N < .00001$ for $N \geq 47$, $r''_N < .00001$ for $N \geq 23$, $r'''_N < .00001$ for $N \geq 16$, and $r^{(4)}_N < .00001$ for $N \geq 13$. Using the bounds on r'''_N, $\sum_{n=1}^{\infty} \frac{1}{n^3}$ lies between $s'''_{20} - .00001 = 1.202044\ldots$ and $s'''_{20} + .00001 = 1.202064\ldots$. We also have $r^{(4)}_N < .000001$ for $N \geq 19$, so $\sum_{n=1}^{\infty} \frac{1}{n^3}$ lies between $s^{(4)}_{20} - .000001 = 1.2020555\ldots$ and $s^{(4)}_{20} + .000001 = 1.2020575\ldots$.

If we accelerate $\sum_{n=1}^{\infty} \frac{1}{n^3}$ k times, for $k \geq 0$, then we get

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = \sum_{j=1}^{k} \frac{c_j}{(j+1)(j+1)!} + \sum_{n=1}^{\infty} \frac{c_{k+1}n + (k+1)!}{n^3(n+1)\cdots(n+k+1)},$$

where the first sum on the right is 0 at $k = 0$ and the integers $c_k = 1, 3, 11, 50, 274, \ldots$ are determined by the recursive relation $c_1 = 1$ and $c_k = kc_{k-1} + (k-1)!$ for $k \geq 2$. These are the (unsigned) Stirling numbers of the first kind that count the number of permutations of the set $\{1, \ldots, k+1\}$ having 2 disjoint cycles.

3. Alternating series: Euler's transformation

The Leibniz series

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \frac{1}{19} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1},$$

which equivalently says

$$\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \frac{4}{11} + \frac{4}{13} - \frac{4}{15} + \frac{4}{17} - \frac{4}{19} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n 4}{2n+1}, \tag{3.1}$$

converges very slowly. For example, the 100th partial sum of the series in (3.1) is $3.151\ldots$, which is accurate to only one digit past the decimal point.

We will describe a method due to Euler for accelerating the convergence of alternating series[2], and illustrate it for both (3.1) and the alternating harmonic series

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}. \tag{3.2}$$

Euler's basic idea is that a convergent alternating series

$$S = a_0 - a_1 + a_2 - a_3 + a_4 - a_5 + \cdots \tag{3.3}$$

can be rewritten as

$$S = \frac{a_0}{2} + \left(\frac{a_0}{2} - \frac{a_1}{2}\right) - \left(\frac{a_1}{2} - \frac{a_2}{2}\right) + \left(\frac{a_2}{2} - \frac{a_3}{2}\right) - \left(\frac{a_3}{2} - \frac{a_4}{2}\right) + \left(\frac{a_4}{2} - \frac{a_5}{2}\right) - \cdots, \tag{3.4}$$

where each term of the original series is split in half and combined with half of the adjacent terms on both sides of the original series (except the first term a_0, where a single $a_0/2$ is left on its own). The *order* of addition has not changed in passing from (3.3) to (3.4), so the value of the series does not change. The significance of (3.4) is that the terms $a_n/2 - a_{n+1}/2$ in this new series may have a faster decay rate than the original terms a_n, and applying this transformation multiple times can accelerate the convergence in an impressive way.

Example 3.1. Applying (3.3) $\rightsquigarrow$ (3.4) to (3.1) turns this series into

$$\begin{aligned}\pi &= 2 + \left(2 - \frac{2}{3}\right) - \left(\frac{2}{3} - \frac{2}{5}\right) + \left(\frac{2}{5} - \frac{2}{7}\right) - \left(\frac{2}{7} - \frac{2}{9}\right) + \left(\frac{2}{9} - \frac{2}{11}\right) - \left(\frac{2}{11} - \frac{2}{13}\right) + \cdots \\ &= 2 + \frac{4}{1\cdot 3} - \frac{4}{3\cdot 5} + \frac{4}{5\cdot 7} - \frac{4}{7\cdot 9} + \frac{4}{9\cdot 11} - \frac{4}{11\cdot 13} + \cdots.\end{aligned}$$

We have changed (3.1) into

$$\pi = 2 + \sum_{n=0}^{\infty} \frac{(-1)^n 4}{(2n+1)(2n+3)} \tag{3.5}$$

by replacing $a_n = \dfrac{4}{2n+1}$ in (3.3) with

$$a'_n = \frac{a_n}{2} - \frac{a_{n+1}}{2} = \frac{2}{2n+1} - \frac{2}{2n+3} = \frac{2(2n+3) - 2(2n+1)}{(2n+1)(2n+3)} = \frac{4}{(2n+1)(2n+3)}. \tag{3.6}$$

[2] Euler gives a brief account of accelerating the series in our Examples 3.1 and 3.2 in [2, Chap. 1, Part II] (pp. 236-237 of the original Latin and pp. 399-400 in the English translation). See also [4, Sect. 35B].

Now view the alternating series in (3.5) as an instance of (3.3) and transform it into (3.4):

$$\begin{aligned}\pi &= 2+\frac{4}{1\cdot 3}-\frac{4}{3\cdot 5}+\frac{4}{5\cdot 7}-\frac{4}{7\cdot 9}+\frac{4}{9\cdot 11}-\frac{4}{11\cdot 13}+\cdots\\ &= 2+\frac{2}{1\cdot 3}+\left(\frac{2}{1\cdot 3}-\frac{2}{3\cdot 5}\right)-\left(\frac{2}{3\cdot 5}-\frac{2}{5\cdot 7}\right)+\left(\frac{2}{5\cdot 7}-\frac{2}{7\cdot 9}\right)-\left(\frac{2}{7\cdot 9}-\frac{2}{9\cdot 11}\right)+\cdots\\ &= 2+\frac{2}{3}+\frac{8}{1\cdot 3\cdot 5}-\frac{8}{3\cdot 5\cdot 7}+\frac{8}{5\cdot 7\cdot 9}-\frac{8}{7\cdot 9\cdot 11}+\cdots\end{aligned}$$

We have changed (3.5) into

$$\pi = 2+\frac{2}{3}+\sum_{n=0}^{\infty}\frac{(-1)^n 8}{(2n+1)(2n+3)(2n+5)} \tag{3.7}$$

by replacing a_n' in (3.6) with

$$a_n'' = \frac{a_n'}{2}-\frac{a_{n+1}'}{2} = \frac{4}{(2n+1)(2n+3)}-\frac{4}{(2n+3)(2n+5)} = \frac{8}{(2n+1)(2n+3)(2n+5)}. \tag{3.8}$$

Next view the alternating series in (3.7) as (3.3) and transform it into (3.4):

$$\begin{aligned}\pi &= 2+\frac{2}{3}+\frac{8}{1\cdot 3\cdot 5}-\frac{8}{3\cdot 5\cdot 7}+\frac{8}{5\cdot 7\cdot 9}-\frac{8}{7\cdot 9\cdot 11}+\cdots\\ &= 2+\frac{2}{3}+\frac{4}{15}+\left(\frac{4}{1\cdot 3\cdot 5}-\frac{4}{3\cdot 5\cdot 7}\right)-\left(\frac{4}{3\cdot 5\cdot 7}-\frac{4}{5\cdot 7\cdot 11}\right)+\cdots\\ &= 2+\frac{2}{3}+\frac{4}{15}+\frac{24}{1\cdot 3\cdot 5\cdot 7}-\frac{24}{3\cdot 5\cdot 7\cdot 9}+\cdots.\end{aligned}$$

We have changed (3.7) into

$$\pi = 2+\frac{2}{3}+\frac{4}{15}+\sum_{n=0}^{\infty}\frac{(-1)^n 24}{(2n+1)(2n+3)(2n+5)(2n+7)} \tag{3.9}$$

by replacing a_n'' in (3.8) with

$$a_n''' = \frac{a_n''}{2}-\frac{a_{n+1}''}{2} = \frac{24}{(2n+1)(2n+3)(2n+5)(2n+7)}. \tag{3.10}$$

Applying this process two more times, we get

$$\pi = 2+\frac{2}{3}+\frac{4}{15}+\frac{12}{105}+\sum_{n=0}^{\infty}\frac{(-1)^n 96}{(2n+1)(2n+3)(2n+5)(2n+7)(2n+9)} \tag{3.11}$$

and

$$\pi = 2+\frac{2}{3}+\frac{4}{15}+\frac{12}{105}+\frac{48}{945}+\sum_{n=0}^{\infty}\frac{(-1)^n 480}{(2n+1)(2n+3)(2n+5)(2n+7)(2n+9)(2n+11)}. \tag{3.12}$$

We now have six partial sums that tend to π:

$$
\begin{aligned}
s_N &= \sum_{n=0}^{N} \frac{(-1)^n 4}{2n+1}, \\
s'_N &= 2 + \sum_{n=0}^{N} \frac{(-1)^n 4}{(2n+1)(2n+3)} \\
s''_N &= 2 + \frac{2}{3} + \sum_{n=0}^{N} \frac{(-1)^n 8}{(2n+1)(2n+3)(2n+5)} \\
s'''_N &= 2 + \frac{2}{3} + \frac{4}{15} + \sum_{n=0}^{N} \frac{(-1)^n 24}{(2n+1)(2n+3)(2n+5)(2n+7)} \\
s^{(4)}_N &= 2 + \frac{2}{3} + \frac{4}{15} + \frac{12}{105} + \sum_{n=0}^{N} \frac{(-1)^n 96}{(2n+1)(2n+3)(2n+5)(2n+7)(2n+9)} \\
s^{(5)}_N &= 2 + \frac{2}{3} + \frac{4}{15} + \frac{12}{105} + \frac{48}{945} + \sum_{n=0}^{N} \frac{(-1)^n 480}{(2n+1)(2n+3)(2n+5)(2n+7)(2n+9)(2n+11)}.
\end{aligned}
$$

The table below lists these partial sums at $N = 10$, 20, 25, 50, 100, and 1000 truncated (not rounded) to 8 digits after the decimal point. While s_{10} is only accurate to one digit, $s^{(5)}_{10}$ is accurate to 6 digits. While s_{100} is only accurate to two digits, $s^{(5)}_{100}$ is accurate to 11 digits (the 9th and 10th digits after the decimal point are not in the table).

N	10	20	25	50	100	1000
s_N	3.23231580	3.18918478	3.10314531	3.16119861	3.15149340	3.14259165
s'_N	3.14535928	3.14267315	3.14088116	3.14178113	3.14164118	3.14159315
s''_N	3.14188102	3.14163956	3.14156726	3.14159620	3.14159312	3.14159265
s'''_N	3.14162337	3.14159558	3.14159134	3.14159275	3.14159266	3.14159265
$s^{(4)}_N$	3.14159672	3.14159288	3.14159256	3.14159265	3.14159265	3.14159265
$s^{(5)}_N$	3.14159328	3.14159267	3.14159264	3.14159265	3.14159265	3.14159265

The reason accelerated series converge faster is that their terms decay to 0 at ever faster rates. Terms in the successive series for π – (3.1), (3.5), (3.7), (3.9), (3.11), and (3.12) – decay as follows:

$$
\begin{aligned}
a_n &= \frac{4}{2n+1} \sim \frac{2}{n}, \\
a'_n &= \frac{4}{(2n+1)(2n+3)} \sim \frac{1}{n^2}, \\
a''_n &= \frac{8}{(2n+1)(2n+3)(2n+5)} \sim \frac{1}{n^3}, \\
a'''_n &= \frac{24}{(2n+1)(2n+3)(2n+5)(2n+7)} \sim \frac{3}{2n^4}, \\
a^{(4)}_n &= \frac{96}{(2n+1)(2n+3)(2n+5)(2n+7)(2n+9)} \sim \frac{3}{n^5}, \\
a^{(5)}_n &= \frac{480}{(2n+1)(2n+3)(2n+5)(2n+7)(2n+9)(2n+11)} \sim \frac{15}{2n^6}.
\end{aligned}
$$

In general, after applying k series accelerations to (3.1) we have

$$\pi = \sum_{j=0}^{k-1} \frac{2(j!)}{1\cdot 3\cdot 5\cdots(2j+1)} + \sum_{n=0}^{\infty} \frac{(-1)^n 4(k!)}{(2n+1)(2n+3)\cdots(2n+2k+1)}, \tag{3.13}$$

where the first sum in (3.13) is 0 for $k = 0$. This formula at $k = 0$, 1, 2, 3, 4, and 5 is (3.1), (3.5), (3.7), (3.9), (3.11), and (3.12) respectively, and for each k the magnitude of the nth term in the alternating series in (3.13) decays like $1/n^{k+1}$ up to a scaling factor: as $n \to \infty$,

$$\frac{4(k!)}{(2n+1)(2n+3)\cdots(2n+2k+1)} \sim \frac{4(k!)}{(2n)^{k+1}} = \frac{k!/2^{k-1}}{n^{k+1}}.$$

In an answer on https://math.stackexchange.com/questions/1702694/why-is-the-leibniz-method-for-approximating-pi-so-inefficient the series (3.1) is accelerated 24 times.

Error bounds on the remainder for each series for π can be obtained from the alternating series test: the absolute value of the first omitted term is a bound. Writing $r_N^{(i)} = \pi - s_N^{(i)}$,

$$|r_N| < \frac{4}{2(N+1)} < \frac{2}{N}, \quad |r'_N| < \frac{4}{4N^2} = \frac{1}{N^2}, \quad |r''_N| < \frac{8}{8N^3} = \frac{1}{N^3},$$
$$|r'''_N| < \frac{24}{16N^4} = \frac{3}{2N^4}, \quad |r_N^{(4)}| < \frac{96}{32N^5} = \frac{3}{N^5}, \quad |r_N^{(5)}| < \frac{480}{64N^6} = \frac{15}{2N^6}.$$

For example, $|r_N^{(4)}| < .000001$ if $3/N^5 < .000001$, which is the same as $N \geq 20$. Thus π is between $s_{20}^{(4)} - .000001 = 3.141591\ldots$ and $s_{20}^{(4)} + .000001 = 3.141593\ldots$.

Example 3.2. Now we turn to the alternating harmonic series (3.2), and will be more brief than we were with the series for π. Write (3.2) as $a_1 - a_2 + a_3 - a_4 + \cdots = \sum_{n=1}^{\infty}(-1)^{n-1}a_n$, where $a_n = \frac{1}{n}$. Accelerating (3.2) once turns that series into

$$\frac{a_1}{2} + \sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{a_n}{2} - \frac{a_{n+1}}{2}\right) = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n(n+1)} = \frac{1}{2} + \frac{1}{4} - \frac{1}{12} + \frac{1}{24} - \frac{1}{40} + \cdots. \tag{3.14}$$

Accelerating (3.14) makes it

$$\frac{1}{2}+\frac{1}{8}+\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{1}{4n(n+1)} - \frac{1}{4(n+1)(n+2)}\right) = \frac{1}{2}+\frac{1}{8}+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n(n+1)(n+2)} \tag{3.15}$$

and the reader should check as an exercise that the next few accelerations are

$$\frac{1}{2} + \frac{1}{8} + \frac{1}{24} + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}3}{4n(n+1)(n+2)(n+3)}, \tag{3.16}$$

$$\frac{1}{2} + \frac{1}{8} + \frac{1}{24} + \frac{1}{64} + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}3}{2n(n+1)(n+2)(n+3)(n+4)}, \tag{3.17}$$

and

$$\frac{1}{2} + \frac{1}{8} + \frac{1}{24} + \frac{1}{64} + \frac{1}{160} + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}15}{4n(n+1)(n+2)(n+3)(n+4)(n+5)}. \tag{3.18}$$

In the table below we list partial sums of (3.2) and its accelerated forms (3.14), (3.15), (3.16), (3.17), and (3.18). The notation $s_N^{(i)}$ in the first column is, by analogy with the series for π, the ith accelerated form of (3.2), for $0 \leq i \leq 5$, with the sum running up to $n = N$.

N	10	20	25	50	100	1000
s_N	.64563492	.66877140	.71274749	.68324716	.68817217	.69264743
s'_N	.69108946	.69258092	.69351673	.69305108	.69312267	.69314693
s''_N	.69298340	.69312205	.69316060	.69314535	.69314694	.69314718
s'''_N	.69312909	.69314557	.69314788	.69314712	.69314717	.69314718
$s^{(4)}_N$	.69314470	.69314705	.69314722	.69314717	.69314718	.69314718
$s^{(5)}_N$	.69314678	.69314716	.69314718	.69314718	.69314718	.69314718

Since all these series are alternating, we can bound remainders using the first missing term. For example, from (3.18), $|r^{(5)}_{20}| \leq 15/(4(21^6)) \approx 4.37/10^8$, so from the value of $s^{(5)}_{20}$ we know (3.2) equals $.693147\ldots$. By comparison, the 1000th partial sum of (3.2) is accurate to just two digits.

Generalizing the series (3.14)–(3.18), after k accelerations (3.2) becomes

$$\sum_{j=1}^{k} \frac{1}{2^j j} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} k!}{2^k n(n+1)\cdots(n+k)}, \tag{3.19}$$

where the first sum in (3.19) is 0 when $k = 0$, and for each k the nth term in the alternating series in (3.19) decays like $1/n^{k+1}$ up to a scaling factor: as $n \to \infty$,

$$\frac{k!}{2^k n(n+1)\cdots(n+k)} \sim \frac{k!/2^k}{n^{k+1}}$$

We can describe the result of applying Euler's transformation k times in general by using the notation of the difference calculus. For a sequence $a = (a_0, a_1, a_2, \ldots)$, its *first discrete difference* is the sequence $\Delta a = (a_1 - a_0, a_2 - a_1, a_3 - a_2, \ldots)$, so $(\Delta a)(n) = a_{n+1} - a_n$. The second discrete difference of the original sequence is $\Delta^2 a = \Delta(\Delta a)$, which starts out as

$$((\Delta a)(1) - (\Delta a)(0), (\Delta a)(2) - (\Delta a)(1), \ldots) = (a_2 - 2a_1 + a_0, a_3 - 2a_2 + a_1, \ldots)$$

and in general the kth discrete difference of the original sequence is $\Delta^k a = \Delta(\Delta^{k-1} a)$. The formula $(\Delta^2 a)(n) = a_{n+2} - 2a_{n+1} + a_n$ suggests a connection with binomial coefficients using alternating signs. In general

$$(\Delta^k a)(n) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} a(n+j).$$

In particular, $(\Delta^k a)(0) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} a_j = a_k - k a_{k-1} + \cdots + (-1)^k a_0$.

With this notation, Euler's transformation in (3.4) consists of rewriting $\sum_{n=0}^{\infty} (-1)^n a_n$ as

$$\frac{a_0}{2} + \sum_{n=0}^{\infty} (-1)^n \frac{a_n - a_{n+1}}{2} = \frac{a_0}{2} - \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n (a_{n+1} - a_n) = \frac{a_0}{2} - \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n (\Delta a)(n). \tag{3.20}$$

Apply Euler's transformation to the new series on the right:

$$\sum_{n=0}^{\infty} (-1)^n (\Delta a)(n) = \frac{(\Delta a)(0)}{2} - \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n (\Delta(\Delta a))(n) = \frac{a_1 - a_0}{2} - \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n (\Delta^2 a)(n),$$

and feeding this into (3.20) shows $\sum_{n=0}^{\infty}(-1)^n a_n$ is

$$\frac{a_0}{2}-\frac{1}{2}\left(\frac{a_1-a_0}{2}-\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(\Delta^2 a)(n)\right)=\frac{a_0}{2}-\frac{(\Delta a)(1)}{4}+\frac{1}{4}\sum_{n=0}^{\infty}(-1)^n(\Delta^2 a)(n).$$

In general, applying Euler's transformation k times leads to

$$\sum_{n=0}^{\infty}a_n(-1)^n=\sum_{j=0}^{k-1}\frac{(-1)^j}{2^{j+1}}(\Delta^j a)(0)+\left(\frac{-1}{2}\right)^k\sum_{n=0}^{\infty}(-1)^n(\Delta^k a)(n), \tag{3.21}$$

where the first (finite) series on the right is 0 at $k=0$.

Remark 3.3. While we are interested in examples where Euler's transformation speeds up convergence, it does not always have such an effect. For example, if $a_n=r^n$ with $|r|<1$ then $(\Delta a)(n)=r^{n+1}-r^n=(r-1)r^n=(r-1)a_n$, so Euler's transformation on a geometric series leads to no improvement:

$$\sum_{n=0}^{\infty}(-1)^n r^n=\frac{1}{2}-\frac{r-1}{2}\sum_{n=0}^{\infty}(-1)^n r^n.$$

A version of Euler's transformation can be applied to any convergent series that looks like a power series $\sum_{n=0}^{\infty}a_n c^n$ for a c in $[-1,1)$, not just when $c=-1$:

$$\begin{aligned}
\sum_{n=0}^{\infty}a_n c^n &= a_0+a_1c+a_2c^2+a_3c^3+\cdots\\
&= a_0\frac{1-c}{1-c}+a_1\frac{c-c^2}{1-c}+a_2\frac{c^2-c^3}{1-c}+a_3\frac{c^3-c^4}{1-c}+\cdots\\
&= \frac{a_0}{1-c}+\frac{-a_0+a_1}{1-c}c+\frac{-a_1+a_2}{1-c}c^2+\frac{-a_2+a_3}{1-c}c^3+\cdots\\
&= \frac{a_0}{1-c}+\frac{a_1-a_0}{1-c}c+\frac{a_2-a_1}{1-c}c^2+\frac{a_3-a_2}{1-c}c^3+\cdots\\
&= \frac{a_0}{1-c}+\frac{c}{1-c}\sum_{n=0}^{\infty}(a_{n+1}-a_n)c^n\\
&= \frac{a_0}{1-c}+\frac{c}{1-c}\sum_{n=0}^{\infty}(\Delta a)(n)c^n.
\end{aligned}$$

When $c=-1$ this is (3.20), and in case the result seems like a trick it could also be derived using summation by parts with $u_n=a_n$ and $v_n=c^{n+1}/(c-1)$. Repeating this process k times for $k\geq 0$,

$$\sum_{n=0}^{\infty}a_n c^n=\sum_{j=0}^{k-1}\frac{c^j}{(1-c)^{j+1}}(\Delta^j a)(0)+\frac{c^k}{(1-c)^k}\sum_{n=0}^{\infty}(\Delta^k a)(n)c^n,$$

where the first sum on the right is 0 at $k=0$. At $c=-1$ the above formula is (3.21).

For more on this, see [3] and [4, Sect. 33, 35], but watch out: in [4], $(\Delta a)(n)=a_n-a_{n+1}$. That is the negative of our convention, so Δ^n in [4] is our $(-1)^n\Delta^n$.

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