THE GAUSSIAN INTEGRAL
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Let
\[ I = \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} \, dx, \quad J = \int_{0}^{\infty} e^{-x^2} \, dx, \quad K = \int_{-\infty}^{\infty} e^{-\pi x^2} \, dx. \]
These numbers are positive, and \( J = I/(2\sqrt{2}) \) and \( K = I/\sqrt{2\pi} \).

**Theorem.** With notation as above, \( I = \sqrt{2\pi} \), or equivalently \( J = \sqrt{\pi/2} \), or equivalently \( K = 1 \).

We will give multiple proofs of this result. (Other lists of proofs are in [3] and [8].) The theorem is subtle because there is no simple antiderivative for \( e^{-\frac{1}{2}x^2} \) (or \( e^{-x^2} \) or \( e^{-\pi x^2} \)). For comparison, \( \int_{0}^{\infty} xe^{-\frac{1}{2}x^2} \, dx \) can be computed using the antiderivative \(-e^{-\frac{1}{2}x^2}\): this integral is 1.

1. **First Proof: Polar coordinates**

The most widely known proof uses multivariable calculus: express \( J^2 \) as a double integral and then pass to polar coordinates:

\[ J^2 = \int_{0}^{\infty} e^{-x^2} \, dx \int_{0}^{\infty} e^{-y^2} \, dy = \int_{0}^{\infty} \int_{0}^{\infty} e^{-(x^2+y^2)} \, dx \, dy. \]

This is a double integral over the first quadrant, which we will compute by using polar coordinates. In polar coordinates, the first quadrant is \( \{(r, \theta) : r \geq 0 \text{ and } 0 \leq \theta \leq \pi/2\} \). Writing \( x^2 + y^2 = r^2 \) and \( dx \, dy = r \, dr \, d\theta \),

\[
\begin{align*}
J^2 &= \int_{0}^{\pi/2} \int_{0}^{\infty} e^{-r^2} \, r \, dr \, d\theta \\
&= \int_{0}^{\infty} re^{-r^2} \, dr \cdot \int_{0}^{\pi/2} \, d\theta \\
&= -\frac{1}{2} e^{-r^2} \bigg|_{0}^{\infty} \cdot \frac{\pi}{2} \\
&= \frac{1}{2} \cdot \frac{\pi}{2} \\
&= \frac{\pi}{4}.
\end{align*}
\]

Taking square roots, \( J = \frac{\sqrt{\pi}}{2} \). This method is due to Poisson [8, p. 3].

2. **Second Proof: Another change of variables**

Our next proof uses another change of variables to compute \( J^2 \), but this will only rely on single-variable calculus. As before, we have

\[ J^2 = \int_{0}^{\infty} \int_{0}^{\infty} e^{-(x^2+y^2)} \, dx \, dy, \]
but instead of using polar coordinates we make a change of variables \( x = yt \) with \( dx = y \, dt \), so
\[
J^2 = \int_0^\infty \int_0^\infty e^{-y^2(t^2+1)} y \, dt \, dy = \int_0^\infty \left( \int_0^\infty ye^{-y^2(t^2+1)} \, dy \right) \, dt.
\]
Since \( \int_0^\infty ye^{-ay^2} \, dy = \frac{1}{2a} \) for \( a > 0 \), we have
\[
J^2 = \int_0^\infty \frac{dt}{2(t^2 + 1)} = \frac{1}{2} \cdot \pi = \frac{\pi}{4},
\]
so \( J = \sqrt{\pi}/2 \). This approach is due to Laplace [6, pp. 94–96] and historically precedes the more familiar technique in the first proof above. We will see in our seventh proof that this was not Laplace’s first method.

3. Third Proof: Differentiating under the integral sign

For \( t > 0 \), set
\[
A(t) = \left( \int_0^t e^{-x^2} \, dx \right)^2.
\]
The integral we want to calculate is \( A(\infty) = J^2 \) and then take a square root.

Differentiating \( A(t) \) with respect to \( t \),
\[
A'(t) = 2 \int_0^t e^{-x^2} \, dx \cdot e^{-t^2} = 2e^{-t^2} \int_0^t e^{-x^2} \, dx.
\]
Let \( x = ty \), so
\[
A'(t) = 2e^{-t^2} \int_0^1 te^{-t^2y^2} \, dy = \int_0^1 2t e^{-(1+y^2)t^2} \, dy.
\]
The function under the integral sign is easily antidifferentiated with respect to \( t \):
\[
A'(t) = \int_0^1 \frac{\partial}{\partial t} \frac{e^{-(1+y^2)t^2}}{1 + y^2} \, dy = -\frac{d}{dt} \int_0^1 \frac{e^{-(1+y^2)t^2}}{1 + y^2} \, dy.
\]
Letting
\[
B(t) = \int_0^1 \frac{e^{-t^2(1+x^2)}}{1 + x^2} \, dx,
\]
we have \( A'(t) = -B'(t) \) for all \( t > 0 \), so there is a constant \( C \) such that
\[
(3.1) \quad A(t) = -B(t) + C
\]
for all \( t > 0 \). To find \( C \), we let \( t \to 0^+ \) in (3.1). The left side tends to \( (\int_0^0 e^{-x^2} \, dx)^2 = 0 \) while the right side tends to \( -\int_0^1 \frac{dx}{1 + x^2} + C = -\pi/4 + C \). Thus \( C = \pi/4 \), so (3.1) becomes
\[
\left( \int_0^t e^{-x^2} \, dx \right)^2 = \frac{\pi}{4} - \int_0^1 \frac{e^{-t^2(1+x^2)}}{1 + x^2} \, dx.
\]
Letting \( t \to \infty \) in this equation, we obtain \( J^2 = \pi/4 \), so \( J = \sqrt{\pi}/2 \).

A comparison of this proof with the first proof is in [17].
4. **Fourth Proof: A volume integral**

Our next proof is due to T. P. Jameson [4] and it was rediscovered by A. L. Delgado [2]. Revolve the curve \( z = e^{-\frac{1}{2}x^2} \) in the \( xz \)-plane around the \( z \)-axis to produce the “bell surface” \( z = e^{-\frac{1}{2}(x^2+y^2)} \). See below, where the \( z \)-axis is vertical and passes through the top point, the \( x \)-axis lies just under the surface through the point 0 in front, and the \( y \)-axis lies just under the surface through the point 0 on the left. We will compute the volume \( V \) below the surface and above the \( xy \)-plane in two ways.

![Graph of the bell surface]

First we compute \( V \) by *horizontal slices*, which are discs: \( V = \int_0^1 A(z) \, dz \) where \( A(z) \) is the area of the disc formed by slicing the surface at height \( z \). Writing the radius of the disc at height \( z \) as \( r(z) \), \( A(z) = \pi r(z)^2 \). To compute \( r(z) \), the surface cuts the \( xz \)-plane at a pair of points \( (x, e^{-\frac{1}{2}x^2}) \) where the height is \( z \), so \( e^{-\frac{1}{2}x^2} = z \). Thus \( x^2 = -2 \ln z \). Since \( x \) is the distance of these points from the \( z \)-axis, \( r(z)^2 = x^2 = -2 \ln z \), so \( A(z) = \pi r(z)^2 = -2\pi \ln z \). Therefore

\[
V = \int_0^1 -2\pi \ln z \, dz = -2\pi \left( z \ln z - z \right) \bigg|_0^1 = -2\pi \left( -1 - \lim_{z \to 0^+} z \ln z \right).
\]

By L’Hospital’s rule, \( \lim_{z \to 0^+} z \ln z = 0 \), so \( V = 2\pi \). (A calculation of \( V \) by shells is in [10].)

Next we compute the volume by *vertical slices* in planes \( x = \) constant. Vertical slices are scaled bell curves: look at the black contour lines in the picture. The equation of the bell curve along the
top of the vertical slice with x-coordinate $x$ is $z = e^{-\frac{1}{2}(x^2+y^2)}$, where $y$ varies and $x$ is fixed. Then

$$V = \int_{-\infty}^{\infty} A(x) \, dx,$$

where $A(x)$ is the area of the $x$-slice:

$$A(x) = \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+y^2)} \, dy = e^{-\frac{1}{2}x^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} \, dy = e^{-\frac{1}{2}x^2} I.$$

Thus $V = \int_{-\infty}^{\infty} A(x) \, dx = \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} I \, dx = I \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} \, dx = I^2.$

Comparing the two formulas for $V$, we have $2\pi = I^2$, so $I = \sqrt{2\pi}$.

5. Fifth Proof: The $\Gamma$-function

For any integer $n \geq 0$, we have $n! = \int_{0}^{\infty} t^n e^{-t} \, dt$. For $x > 0$ we define

$$\Gamma(x) = \int_{0}^{\infty} t^{x-1} e^{-t} \, dt,$$

so $\Gamma(n) = (n-1)!$ when $n \geq 1$. Using integration by parts, $\Gamma(x+1) = x\Gamma(x)$. One of the basic properties of the $\Gamma$-function [13, pp. 193–194] is

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_{0}^{1} t^{x-1}(1-t)^{y-1} \, dt.$$

Set $x = y = \frac{1}{2}$:

$$\Gamma\left(\frac{1}{2}\right)^2 = \int_{0}^{1} \frac{dt}{\sqrt{t(1-t)}}.$$

Note

$$\Gamma\left(\frac{1}{2}\right) = \int_{0}^{\infty} \sqrt{t}e^{-t} \frac{dt}{t} = \int_{0}^{\infty} \frac{e^{-t}}{\sqrt{t}} \, dt = \int_{0}^{\infty} \frac{e^{-x^2}}{x} \, dx = 2 \int_{0}^{\infty} e^{-x^2} \, dx = 2J,$$

so $4J^2 = \int_{0}^{1} \frac{dt}{\sqrt{t(1-t)}}$. With the substitution $t = \sin^2 \theta$,

$$4J^2 = \int_{0}^{\pi/2} 2 \sin \theta \cos \theta \frac{d\theta}{\sin \theta \cos \theta} = 2 \frac{\pi}{2} = \pi,$$

so $J = \sqrt{\pi}/2$. Equivalently, $\Gamma(1/2) = \sqrt{\pi}$. Any method that proves $\Gamma(1/2) = \sqrt{\pi}$ is also a method that calculates $\int_{0}^{\infty} e^{-x^2} \, dx$.

6. Sixth Proof: Asymptotic Estimates

We will show $J = \sqrt{\pi}/2$ by a technique whose steps are based on [14, p. 371].

For $x \geq 0$, power series expansions show $1 + x \leq e^x \leq 1/(1 - x)$. Reciprocating and replacing $x$ with $x^2$, we get

$$1 - x^2 \leq e^{-x^2} \leq \frac{1}{1 + x^2}.$$

for all $x \in \mathbb{R}$.

For any positive integer $n$, raise the terms in (6.1) to the $n$th power and integrate from 0 to 1:

$$\int_{0}^{1} (1 - x^2)^n \, dx \leq \int_{0}^{1} e^{-nx^2} \, dx \leq \int_{0}^{1} \frac{dx}{(1 + x^2)^n}.$$
Under the changes of variables \( x = \sin \theta \) on the left, \( x = y/\sqrt{n} \) in the middle, and \( x = \tan \theta \) on the right,

\[
\int_0^{\pi/2} (\cos \theta)^{2n+1} \, d\theta \leq \frac{1}{\sqrt{n}} \int_0^{\sqrt{n}} e^{-y^2} \, dy \leq \int_0^{\pi/4} (\cos \theta)^{2n-2} \, d\theta.
\]

Set \( I_k = \int_0^{\pi/2} (\cos \theta)^k \, d\theta \), so \( I_0 = \pi/2 \), \( I_1 = 1 \), and (6.2) implies

\[
\sqrt{n}I_{2n+1} \leq \int_0^{\sqrt{n}} e^{-y^2} \, dy \leq \sqrt{n}I_{2n-2}.
\]

We will show that as \( k \to \infty \), \( kI_2^k \to \pi/2 \). Then

\[
\sqrt{n}I_{2n+1} = \frac{\sqrt{n}}{\sqrt{2n+1}} \sqrt{2n+1}I_{2n+1} \to \frac{\sqrt{\pi}}{2} = \frac{\sqrt{\pi}}{2}
\]

and

\[
\sqrt{n}I_{2n-2} = \frac{\sqrt{n}}{\sqrt{2n-2}} \sqrt{2n-2}I_{2n-2} \to \frac{\sqrt{\pi}}{2} = \frac{\sqrt{\pi}}{2},
\]

so by (6.3) \( \int_0^{\sqrt{n}} e^{-y^2} \, dy \to \sqrt{\pi}/2 \). Thus \( J = \sqrt{\pi}/2 \).

To show \( kI_2^k \to \pi/2 \), first we compute several values of \( I_k \) explicitly by a recursion. Using integration by parts,

\[
I_k = \int_0^{\pi/2} (\cos \theta)^k \, d\theta = \int_0^{\pi/2} (\cos \theta)^{k-1} \cos \theta \, d\theta = (k-1)(I_{k-2} - I_k),
\]

so

\[
I_k = \frac{k-1}{k}I_{k-2}.
\]

Using (6.4) and the initial values \( I_0 = \pi/2 \) and \( I_1 = 1 \), the first few values of \( I_k \) are computed and listed in Table 1.

<table>
<thead>
<tr>
<th>( k )</th>
<th>( I_k )</th>
<th>( k )</th>
<th>( I_k )</th>
</tr>
</thead>
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<td>0</td>
<td>( \pi/2 )</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>((1/2)(\pi/2))</td>
<td>3</td>
<td>2/3</td>
</tr>
<tr>
<td>4</td>
<td>((3/8)(\pi/2))</td>
<td>5</td>
<td>8/15</td>
</tr>
<tr>
<td>6</td>
<td>((15/48)(\pi/2))</td>
<td>7</td>
<td>48/105</td>
</tr>
</tbody>
</table>

Table 1.

From Table 1 we see that

\[
I_{2n}I_{2n+1} = \frac{1}{2n+1}\frac{\pi}{2}
\]

for \( 0 \leq n \leq 3 \), and this can be proved for all \( n \) by induction using (6.4). Since \( 0 \leq \cos \theta \leq 1 \) for \( \theta \in [0, \pi/2] \), we have \( I_k \leq I_{k-1} \leq I_{k-2} = \frac{k}{k-1}I_k \) by (6.4), so \( I_{k-1} \sim I_k \) as \( k \to \infty \). Therefore (6.5) implies

\[
I_{2n}^2 \sim \frac{1}{2n} \frac{\pi}{2} \implies (2n)I_{2n}^2 \to \frac{\pi}{2}
\]

as \( n \to \infty \). Then

\[
(2n+1)I_{2n+1}^2 \sim (2n)I_{2n}^2 \to \frac{\pi}{2}
\]

as \( n \to \infty \), so \( kI_k^2 \to \pi/2 \) as \( k \to \infty \). This completes our proof that \( J = \sqrt{\pi}/2 \).
Remark 6.1. This proof is closely related to the fifth proof using the \( \Gamma \)-function. Indeed, by \((5.1)\)
\[
\frac{\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{k+2}{2} + \frac{1}{2}\right)} = \int_{0}^{1} t^{(k+1)/2+1}(1 - t)^{1/2-1} \, dt,
\]
and with the change of variables \( t = (\cos \theta)^{2} \) for \( 0 \leq \theta \leq \pi/2 \), the integral on the right is equal to
\[
2 \int_{0}^{\pi/2} (\cos \theta)^{k} \, d\theta = 2I_{k},
\]
so \((6.5)\) is the same as
\[
\pi \frac{\pi}{2} (2n + 1) = \int_{0}^{\pi/2} (\cos \theta)^{k} \, d\theta = 2I_{k},
\]
or equivalently \( \Gamma(1/2)^{2} = \pi \). We saw in the fifth proof that \( \Gamma(1/2) = \sqrt{\pi} \) if and only if \( J = \sqrt{\pi}/2 \).

7. Seventh Proof: The original proof

The original proof that \( J = \sqrt{\pi}/2 \) is due to Laplace \([7]\) in 1774. (An English translation of Laplace’s article is mentioned in the bibliographic citation for \([7]\), with preliminary comments on that article in \([15]\).) He wanted to compute
\[
\int_{0}^{1} \frac{dx}{\sqrt{-\log x}}.
\]
Setting \( y = \sqrt{-\log x} \), this integral is \( 2 \int_{0}^{\infty} e^{-y^{2}} \, dy = 2J \), so we expect \((7.1)\) to be \( \sqrt{\pi} \).

Laplace’s starting point for evaluating \((7.1)\) was a formula of Euler:
\[
\int_{0}^{1} \frac{x^{r}}{\sqrt{1 - x^{2s}}} \, dx = \frac{\pi}{(r+1)/2}.
\]
for positive \( r \) and \( s \). (Laplace himself said this formula held “whatever be” \( r \) or \( s \), but if \( s < 0 \) then the number under the square root is negative.) Accepting \((7.2)\), let \( r \to 0 \) in it to get
\[
\int_{0}^{1} \frac{dx}{\sqrt{1 - x^{2}}} \int_{0}^{1} x^{s} \, dx = \frac{\pi}{s}.
\]
Now let \( s \to 0 \) in \((7.3)\). Then \( 1 - x^{2s} \sim -2s \log x \) by L’Hospital’s rule, so \((7.3)\) becomes
\[
\left( \int_{0}^{1} \frac{dx}{\sqrt{-\log x}} \right)^{2} = \pi.
\]
Thus \((7.1)\) is \( \sqrt{\pi} \).

Euler’s formula \((7.2)\) looks mysterious, but we have met it before. In the formula let \( x^{s} = \cos \theta \) with \( 0 \leq \theta \leq \pi/2 \). Then \( x = (\cos \theta)^{1/s} \), and after some calculations \((7.2)\) turns into
\[
\int_{0}^{\pi/2} (\cos \theta)^{(r+1)/s - 1} \, d\theta \int_{0}^{\pi/2} (\cos \theta)^{(r+1)/s} \, d\theta = \frac{1}{(r+1)/s} \pi.
\]
We used the integral $I_k = \int_0^{\pi/2} (\cos \theta)^k d\theta$ before when $k$ is a nonnegative integer. This notation makes sense when $k$ is any positive real number, and then (7.4) assumes the form $I_\alpha I_{\alpha+1} = \frac{1}{\alpha+1} \frac{\pi}{2}$ for $\alpha = (r+1)/s-1$, which is (6.5) with a possibly nonintegral index. Letting $r = 0$ and $s = 1/(2n+1)$ in (7.4) recovers (6.5). Letting $s \to 0$ in (7.3) corresponds to letting $n \to \infty$ in (6.5), so the 6th proof is in essence a more detailed version of Laplace’s 1774 argument.

8. Eighth Proof: Contour Integration

We will calculate $\int_{-\infty}^{\infty} e^{-x^2/2} dx$ using contour integrals and the residue theorem. However, we can’t just integrate $e^{-z^2/2}$, as this function has no poles. For a long time nobody knew how to handle this integral using contour integration. For instance, in 1914 Watson [10, p. 79] wrote at the end of his book “Cauchy’s theorem cannot be employed to evaluate all definite integrals; thus $\int_0^{\infty} e^{-x^2} dx$ has not been evaluated except by other methods.” In the 1940s several contour integral solutions were published using awkward contours such as parallelograms [9], [11, Sect. 5] (see [1, Exer. 9, p. 113] for a recent appearance). Our approach will follow Kneser [5, p. 121] (see also [12, pp. 413–414] or [13]), using a rectangular contour and the function

$$\frac{e^{-z^2/2}}{1 - e^{-\sqrt{\pi}(1+i)z}}.$$ 

This function comes out of nowhere, so our first task is to motivate the introduction of this function.

We seek a meromorphic function $f(z)$ to integrate around the rectangular contour $\gamma_R$ in the figure below, with vertices at $-R$, $R$, $R + ib$, and $-R + ib$, where $b$ will be fixed and we let $R \to \infty$.

![Contour Diagram](image)

Suppose $f(z) \to 0$ along the right and left sides of $\gamma_R$ uniformly as $R \to \infty$. Then by applying the residue theorem and letting $R \to \infty$, we would obtain (if the integrals converge)

$$\int_{-\infty}^{\infty} f(x) dx + \int_{-\infty}^{\infty} f(x + ib) dx = 2\pi i \sum_a \text{Res}_{z=a} f(z),$$

where the sum is over poles of $f(z)$ with imaginary part between 0 and $b$. This is equivalent to

$$\int_{-\infty}^{\infty} (f(x) - f(x + ib)) dx = 2\pi i \sum_a \text{Res}_{z=a} f(z).$$

Therefore we want $f(z)$ to satisfy

$$f(z) - f(z + ib) = e^{-z^2/2}, \quad (8.1)$$

where $f(z)$ and $b$ need to be determined.

Let’s try $f(z) = e^{-z^2/2}/d(z)$, for an unknown denominator $d(z)$ whose zeros are poles of $f(z)$. We want $f(z)$ to satisfy

$$f(z) - f(z + \tau) = e^{-z^2/2}, \quad (8.2)$$
for some $\tau$ (which will not be purely imaginary, so (8.1) doesn’t quite work, but (8.1) is only motivation). Substituting $e^{-z^2/2}/d(z)$ for $f(z)$ in (8.2) gives us

\[
(8.3) \quad e^{-z^2/2} \left( \frac{1}{d(z)} - \frac{e^{-\tau z - \tau^2/2}}{d(z + \tau)} \right) = e^{-z^2/2}.
\]

Suppose $d(z + \tau) = d(z)$. Then (8.3) implies

\[
d(z) = 1 - e^{-\tau z - \tau^2/2},
\]

and with this definition of $d(z)$, $f(z)$ satisfies (8.2) if and only if $e^\tau = 1$, or equivalently $\tau^2 \in 2\pi i \mathbb{Z}$. The simplest nonzero solution is $\tau = \sqrt{\pi}(1 + i)$. From now on this is the value of $\tau$, so $e^{-\tau^2/2} = e^{-i\pi} = -1$ and then

\[
f(z) = \frac{e^{-z^2/2}}{d(z)} = \frac{e^{-z^2/2}}{1 + e^{-\tau z}},
\]

which is Kneser’s function mentioned earlier. This function satisfies (8.2) and we henceforth ignore the motivation (8.1). Poles of $f(z)$ are at odd integral multiples of $\tau/2$.

We will integrate this $f(z)$ around the rectangular contour $\gamma_R$ below, whose height is $\text{Im}(\tau)$.

The poles of $f(z)$ nearest the origin are plotted in the figure; they lie along the line $y = x$. The only pole of $f(z)$ inside $\gamma_R$ (for $R > \sqrt{\pi}/2$) is at $\tau/2$, so by the residue theorem

\[
\int_{\gamma_R} f(z) \, dz = 2\pi i \text{Res}_{z=\tau/2} f(z) = 2\pi i \frac{e^{-\tau^2/8}}{(-\tau)e^{-\tau^2/2}} = \frac{2\pi i e^{3\tau^2/8}}{-\sqrt{\pi}(1 + i)} = \sqrt{2\pi}.
\]

As $R \to \infty$, the value of $|f(z)|$ tends to 0 uniformly along the left and right sides of $\gamma_R$, so

\[
\sqrt{2\pi} = \int_{-\infty}^{\infty} f(x) \, dx + \int_{\infty + i\sqrt{\pi}}^{-\infty + i\sqrt{\pi}} f(z) \, dz = \int_{-\infty}^{\infty} f(x) \, dx - \int_{-\infty}^{\infty} f(x + i\sqrt{\pi}) \, dx.
\]
In the second integral, write \( i \sqrt{\pi} \) as \( \tau - \pi \) and use (real) translation invariance of \( dx \) to obtain
\[
\sqrt{2\pi} = \int_{-\infty}^{\infty} f(x) \, dx - \int_{-\infty}^{\infty} f(x + \tau) \, dx
= \int_{-\infty}^{\infty} (f(x) - f(x + \tau)) \, dx
= \int_{-\infty}^{\infty} e^{-x^2/2} \, dx \quad \text{by (8.2)}.
\]

9. 

Ninth Proof: Stirling’s Formula

Besides the integral formula \( \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} \, dx = \sqrt{2\pi} \) that we have been discussing, another place in mathematics where \( \sqrt{2\pi} \) appears is in Stirling’s formula:
\[
n! \sim \frac{n^n}{e^n} \sqrt{2\pi n} \quad \text{as } n \to \infty.
\]

In 1730 De Moivre proved \( n! \sim C(n^n/e^n)\sqrt{n} \) for some positive number \( C \) without being able to determine \( C \). Stirling soon thereafter showed \( C = \sqrt{2\pi} \) and wound up having the whole formula named after him. We will show that determining that the constant \( C \) in Stirling’s formula is \( \sqrt{2\pi} \) is equivalent to showing that \( J = \sqrt{\pi}/2 \) (or, equivalently, that \( I = \sqrt{2\pi} \)).

Applying \([6.4]\) repeatedly,
\[
I_{2n} = \frac{2n-1}{2n} I_{2n-2}
= \frac{(2n-1)(2n-3)}{(2n)(2n-2)} I_{2n-4}
\vdots
= \frac{(2n-1)(2n-3)(2n-5)\cdots(5)(3)(1)}{(2n)(2n-2)(2n-4)\cdots(6)(4)(2)} I_0.
\]

Inserting \((2n-2)(2n-4)(2n-6)\cdots(6)(4)(2)\) in the top and bottom,
\[
I_{2n} = \frac{(2n-1)(2n-3)(2n-5)\cdots(6)(4)(2)(1) \pi}{(2n)(2n-2)(2n-4)\cdots(6)(4)(2)^2} \frac{\pi}{2} = \frac{(2n-1)!}{2n^{2n-1}(n-1)!} \frac{\pi}{2}.
\]

Applying De Moivre’s asymptotic formula \( n! \sim C(n/e)^n \sqrt{n} \),
\[
I_{2n} \sim \frac{C((2n-1)/e)^{2n-1} \sqrt{2n-1}}{2n^{2n-1}C((n-1)/e)^{n-1}\sqrt{n-1}} \frac{\pi}{2} = \frac{(2n-1)^{2n-1} \frac{1}{2n-1} \sqrt{2n-1} \frac{\pi}{2}}{2n \cdot 2^{2n-1} \pi Ce(n-1)^{2n-1} \frac{1}{(n-1)!} \sqrt{n-1}}.
\]
as \( n \to \infty \). For any \( a \in \mathbb{R} \), \((1 + a/n)^n \to e^a \) as \( n \to \infty \), so \((n + a)^n \sim e^a n^n \). Substituting this into the above formula with \( a = -1 \) and \( n \) replaced by \( 2n \),
\[
(9.1)
I_{2n} \sim \frac{e^{-1}(2n)^{2n-1} \sqrt{2n} \frac{\pi}{2}}{2n \cdot 2^{2n-1} Ce(e^{-1}n)^2 \frac{1}{\pi} n^2} = \frac{\pi}{C \sqrt{2n}}.
\]

Since \( I_{k-1} \sim I_k \), the outer terms in \([6.3]\) are both asymptotic to \( \sqrt{n} I_{2n} \sim \pi/(C \sqrt{2}) \) by \([9.1]\). Therefore
\[
\int_0^{\sqrt{n}} e^{-y^2} \, dy \to \frac{\pi}{C \sqrt{2}}
\]
as \( n \to \infty \), so \( J = \pi/(C \sqrt{2}) \). Therefore \( C = \sqrt{2\pi} \) if and only if \( J = \sqrt{\pi}/2 \).
For a continuous function \( f: \mathbb{R} \to \mathbb{C} \) that is rapidly decreasing at \( \pm \infty \), its Fourier transform is the function \( \mathcal{F}f: \mathbb{R} \to \mathbb{C} \) defined by
\[
(\mathcal{F}f)(y) = \int_{-\infty}^{\infty} f(x) e^{-ixy} \, dx.
\]
For example, \((\mathcal{F}f)(0) = \int_{-\infty}^{\infty} f(x) \, dx\).

Here are three properties of the Fourier transform.

- If \( f \) is differentiable, then after using differentiation under the integral sign on the Fourier transform of \( f \) we obtain
  \[
  (\mathcal{F}f)'(y) = \int_{-\infty}^{\infty} -ixf(x) e^{-ixy} \, dx = -i(\mathcal{F}(xf(x)))(y).
  \]
- Using integration by parts on the Fourier transform of \( f \), with \( u = f(x) \) and \( \, dv = e^{-ixy} \, dx \), we obtain
  \[
  \mathcal{F}(f')(y) = iy(\mathcal{F}f)(y).
  \]
- If we apply the Fourier transform twice then we recover the original function up to interior and exterior scaling:
\[
(\mathcal{F}^2 f)(x) = 2\pi f(-x).
\]
Let’s show the appearance of \( 2\pi \) in \((10.1)\) is equivalent to the evaluation of \( I \) as \( \sqrt{2\pi} \).

Fixing \( a > 0 \), set \( f(x) = e^{-ax^2} \), so
\[
f'(x) = -2axf(x).
\]
Applying the Fourier transform to both sides of this equation implies
\[
(\mathcal{F}f)(y) = -2ae^{-\frac{1}{4a}y^2}.
\]
which simplifies to
\[
(\mathcal{F}f)'(y) = -\frac{1}{2a}y(\mathcal{F}f)(y).
\]
The general solution of \( g'(y) = -\frac{1}{2a}yg(y) \) is \( g(y) = Ce^{-y^2/(4a)} \), so
\[
(\mathcal{F}f)(y) = Ce^{-y^2/(4a)}
\]
for some constant \( C \). Letting \( a = \frac{1}{2} \), so \( f(x) = e^{-x^2/2} \), we obtain
\[
(\mathcal{F}f)(y) = Ce^{-y^2/2} = Cf(y).
\]
Setting \( y = 0 \), the left side is \((\mathcal{F}f)(0) = \int_{-\infty}^{\infty} e^{-x^2/2} \, dx = I \), so \( I = C(f(0)) = C \).

Applying the Fourier transform to both sides of the equation \((\mathcal{F}f)(y) = Cf(y)\), we get \( 2\pi f(-x) = C(\mathcal{F}f)(x) = C^2f(x) \). At \( x = 0 \) this becomes \( 2\pi = C^2 \), so \( I = C = \pm \sqrt{2\pi} \). Since \( I > 0 \), the number \( I \) is \( \sqrt{2\pi} \). If we didn’t know the constant on the right side of \((10.1)\) were \( 2\pi \), whatever its value is would wind up being \( C^2 \), so saying \( 2\pi \) appears on the right side of \((10.1)\) is equivalent to saying \( I = \sqrt{2\pi} \).

References


