THE REMAINDER IN TAYLOR SERIES

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1. Introduction

Let $f(x)$ be an infinitely differentiable function on an interval around a number a. On this interval, Taylor's inequality bounds the difference between $f(x)$ and

$$T_{n,a}(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

in terms of the magnitude of the $(n+1)$th derivative of f. Taylor's inequality says that if b is another number in the interval and $|f^{(n+1)}(x)| \leq M$ for all x from a to b, then

$$|f(b) - T_{n,a}(b)| \leq M\frac{|b-a|^{n+1}}{(n+1)!}.$$

We will derive this inequality in two ways, using exact formulas for $f(x) - T_{n,a}(x)$ involving derivatives and involving integrals.

Theorem 1.1 (Differential form of the remainder (Lagrange, 1797))**.** *With notation as above, for all $n \geq 0$*

$$f(b) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(b-a)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}$$

for some c strictly between a and b.

The number c depends on a, b, and n. When $n = 0$ the theorem says $f(b) = f(a) + f'(c)(b-a)$ for some c strictly between a and b, which is the Mean Value Theorem.

Theorem 1.2 (Integral form of the remainder (Cauchy, 1821))**.** *With notation as above, for all $n \geq 0$*

$$f(b) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(b-a)^k + \int_a^b \frac{f^{(n+1)}(t)}{n!}(b-t)^n\,dt.$$

If $n = 0$ this is $f(b) = f(a) + \int_a^b f'(t)\,dt$, which is the Fundamental Theorem of Calculus. Unlike the differential form of the remainder, the integral form of the remainder involves no weird parameters like c.

2. Differential form of the remainder

To prove Theorem 1.1 we will use an extension of Rolle's theorem to higher derivatives.

Theorem 2.1. *Let $F(x)$ be infinitely differentiable on an interval containing the numbers a and b, where $a \neq b$. If $F(a) = F(b)$ and $F^{(j)}(a) = 0$ for $1 \leq j \leq n$, then there is some c strictly between a and b such that $F^{(n+1)}(c) = 0$.*

Proof. By Rolle's theorem for $F(x)$, since $F(a) = F(b)$ we have $F'(c_1) = 0$ for some c_1 strictly between a and b. Then from $F'(a) = 0$ and $F'(c_1) = 0$, by Rolle's theorem for the function $F'(x)$ we have $F''(c_2) = 0$ for some c_2 strictly between a and c_1 (so c_2 is strictly between a and b). Next, from $F''(a) = 0$ and $F''(c_2) = 0$ we have by Rolle's theorem for $F''(x)$ that $F'''(c_3) = 0$ for some c_3 strictly between a and c_2 (so c_3 is strictly between a and b).

Continuing in this way, we eventually get some c_n strictly between a and b such that $F^{(n)}(c_n) = 0$. Combining this with $F^{(n)}(a) = 0$ there is some c_{n+1} strictly between a and c_n (so also strictly between a and b) such that $F^{(n+1)}(c) = 0$. □

Proof of Theorem 1.1. Set

$$E(x) = f(x) - T_{n,a}(x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Due to the choice of coefficients in $T_{n,a}(x)$ we have

$$E(a) = f(a) - f(a) = 0, \quad E'(a) = f'(a) - f'(a) = 0, \quad \ldots, \quad E^{(n)}(a) = f^{(n)}(a) - f^{(n)}(a) = 0.$$

Therefore if we set

$$F(x) = E(x) - \frac{E(b)}{(b-a)^{n+1}}(x-a)^{n+1} \tag{2.1}$$

then we have $F^{(j)}(a) = 0$ for $j = 0, \ldots, n$ and $F(b) = E(b) - E(b) = 0$. By Theorem 2.1, there is some c strictly between a and b such that $F^{(n+1)}(c) = 0$. Since

$$F^{(n+1)}(x) = E^{(n+1)}(x) - \frac{E(b)(n+1)!}{(b-a)^{n+1}} = f^{(n+1)}(x) - \frac{E(b)(n+1)!}{(b-a)^{n+1}},$$

we get

$$0 = F^{(n+1)}(c) = f^{(n+1)}(c) - \frac{E(b)(n+1)!}{(b-a)^{n+1}},$$

so

$$\frac{E(b)}{(b-a)^{n+1}} = \frac{f^{(n+1)}(c)}{(n+1)!}.$$

Substituting this into (2.1),

$$F(x) = E(x) - \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1},$$

so at $x = b$ we get

$$0 = F(b) = E(b) - \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1} = f(b) - T_{n,a}(b) - \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}.$$

□

Taylor's inequality is an immediate consequence of this differential form of the remainder: if $|f^{(n+1)}(x)| \leq M$ for all x from a to b, then $|f^{(n+1)}(c)| \leq M$, so $|f(b) - T_{n,a}(b)| = |f^{(n+1)}(c)(b-a)^{n+1}/(n+1)!| \leq M|b-a|^{n+1}/(n+1)!$.

3. Integral form of the remainder

Proof of Theorem 1.2. Start with the Fundamental Theorem of Calculus in the form

$$f(b) = f(a) + \int_a^b f'(t)\,dt.$$

Apply integration by parts with $u = f'(t)$ and $dv = dt$, so $du = f''(t)\,dt$ and take $v = t - b$ (not $v = t$!) to get

$$\begin{aligned} f(b) &= f(a) + f'(t)(t-b)\Big|_a^b - \int_a^b (t-b)f''(t)\,dt \\ &= f(a) - f'(a)(a-b) - \int_a^b (t-b)f''(t)\,dt \\ &= f(a) + f'(a)(b-a) + \int_a^b (b-t)f''(t)\,dt. \end{aligned}$$

Apply integration by parts again with $u = f''(t)$ and $dv = (b-t)\,dt$, so $du = f'''(t)\,dt$ and take $v = -(b-t)^2/2$. Then

$$\begin{aligned} f(b) &= f(a) + f'(a)(b-a) + \int_a^b (b-t)f''(t)\,dt \\ &= f(a) + f'(a)(b-a) - \frac{f''(t)}{2}(b-t)^2\Big|_a^b + \int_a^b \frac{(b-t)^2}{2} f'''(t)\,dt \\ &= f(a) + f'(a)(b-a) + \frac{f''(a)}{2}(b-a)^2 + \int_a^b \frac{(b-t)^2}{2} f'''(t)\,dt. \end{aligned}$$

After another integration by parts with $u = f'''(t)$ and $dv = \frac{1}{2}(b-t)^2\,dt$ we get

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2}(b-a)^2 + \frac{f'''(a)}{6}(b-a)^3 + \int_a^b \frac{(b-t)^3}{6} f^{(4)}(t)\,dt.$$

After repeated integration by parts we eventually get

$$f(b) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(b-a)^k + \int_a^b \frac{(b-t)^n}{n!} f^{(n+1)}(t)\,dt.$$

□

We will derive Taylor's inequality from Theorem 1.2 in two ways.

Method 1: Assume $|f^{(n+1)}(t)| \leq M$ for all t from a to b. For $a < b$,

$$|f(b) - T_{n,a}(b)| \leq \int_a^b \frac{|b-t|^n}{n!}|f^{(n+1)}(t)|\,dt \leq \int_a^b \frac{(b-t)^n}{n!} M\,dt = M\frac{(b-a)^{n+1}}{(n+1)!},$$

where the last calculation comes from the Fundamental Theorem of Calculus. For $b < a$, we get in a similar way $|f(b) - T_{n,a}(b)| \le M(a-b)^{n+1}/(n+1)!$. Putting both cases together,

$$|f^{(n+1)}(t)| \le M \text{ for } t \text{ from } a \text{ to } b \Longrightarrow |f(b) - T_{n,a}(b)| \le M\frac{|b-a|^{n+1}}{(n+1)!}.$$

Method 2: We make a change of variables in the integral to bypass the need for separate cases as in the first method. The integral is taken from a to b (whether or not $a < b$ or $a > b$), and numbers from a to b can be written in parametric form as $a + (b-a)u$ as u runs from 0 to 1. Therefore with the change of variables $t = a + (b-a)u$ the integral remainder equals

$$\begin{aligned}
\int_a^b \frac{(b-t)^n}{n!} f^{(n+1)}(t)\,dt &= \int_0^1 \frac{((b-a)(1-u))^n}{n!} f^{(n+1)}(a+(b-a)u)(b-a)\,du \\
&= \frac{(b-a)^{n+1}}{n!}\int_0^1 (1-u)^n f^{(n+1)}(a+(b-a)u)\,du,
\end{aligned}$$

so if $|f^{(n+1)}(t)| \le M$ for all t from a to b then the absolute value of the integral remainder is at most

$$\begin{aligned}
\frac{|b-a|^{n+1}}{n!}\int_0^1 (1-u)^n M\,du &= M\frac{|b-a|^{n+1}}{n!}\int_0^1 (1-u)^n\,du \\
&= M\frac{|b-a|^{n+1}}{n!}\frac{1}{n+1} \\
&= M\frac{|b-a|^{n+1}}{(n+1)!}.
\end{aligned}$$

Remark 3.1. Taylor's inequality is not due to Taylor. In fact, Taylor's treatment of power series (in his book *Methodus Incrementorum Directa et Inversa*, written in 1715) was not concerned with justifications of convergence or error estimates, and preceded by almost 80 years the work of Lagrange and by over 100 years the work of Cauchy.