DEFN: Notation will include ω for the nonnegative integers $\{0,1,2,\dots\}$
lower case letters for integers (e.g., $a,b,c,\ i,j,k,\ x,y,z$),
lower case letters for total functions (e.g., f,g,h),
letters for (possibly) partial functions (e.g., $\Phi, \Psi, \Theta, \varphi, \psi, \theta$),
upper case letters for subsets of ω (e.g., A,B,C,X,Y,Z).

DEFN: If φ a partial function, write $\varphi(x)\downarrow$ if $\varphi(x)$ is defined and $\varphi(x)\uparrow$ otherwise. Moreover, write $\varphi(x)\downarrow = y$ if it is defined with value y.

DEFN: Pairing function $\langle \cdot,\cdot \rangle : \omega\times\omega \to \omega$ given by $\frac{1}{2}(x^2+2xy+y^2+3x+y)$. Also Θ.

DEFN: Denote by φ_e the e^{th} partial function. Write $\varphi_{e,s}(x)\downarrow = y$ if $x,y,e < s$ and y is the output of $\varphi_e(x)$ in less than s steps; write $\varphi_{e,s}(x)\uparrow$ if no output in less than s steps.

EXAMPLE: If $\varphi_{e,s}(x)\downarrow = y$, then $\varphi_{e,t}(x)\downarrow = y$ for all $t>s$.
If $\varphi_e(x)\downarrow$, then $(\exists s)[\varphi_{e,s}(x)\downarrow]$.
If $\varphi_e(x)\uparrow$, then $(\forall s)[\varphi_{e,s}(x)\uparrow]$.

DEFN: A set X is computable if and only if

$$\chi_X(x) = \begin{cases} 1 & \text{if } x\in X \\ 0 & \text{otherwise} \end{cases}$$

is a computable function, i.e., $\chi_X = \varphi_e$ for some index e.

DEFN: Denote by K the Halting Problem, i.e., the set $\{x : \varphi_x(x)\downarrow\} = \{x : x \in W_x\}$.

PROP: The set K is computably enumerable but not computable.

PROOF: It is the domain of the partial computable function

$$\psi(x) = \begin{cases} x & \text{if } \varphi_x(x)\downarrow \\ \uparrow & \text{otherwise.} \end{cases}$$

It cannot be computable, else

$$f(x) = \begin{cases} \varphi_x(x)+1 & \text{if } x \in K \\ 0 & \text{if } x \notin K \end{cases}$$

would be computable.

THM: (Normal Form for C.E. Sets) A set A is computably enumerable iff A is Σ^0_1.

PROOF: If A is c.e., then $A = W_e$ for some e. Then $x \in A$ iff $(\exists s)[x \in W_{e,s}]$.

If A is Σ^0_1, then $A = \{x : (\exists y) R(x,y)\}$ for a computable predicate R.

Then $A = \mathrm{dom}\, \psi$ where $\psi = (\mu y) R(x,y)$.

THM: (Complementation) A set A is computable iff A and $\overline{A}$ are computably enumerable.

COR: The set $\overline{K}$ is not computably enumerable.

THM: (Recursion) If f is computable, then there exists an index n such that $\varphi_n = \varphi_{f(n)}$.

Thm: (s-m-n) For every $m,n \geq 1$, there is an injective computable function s^m_n of $m+1$ variables such that

$$\phi^{(n)}_{s^m_n(x,\bar{y})} = \lambda z_1 \dots z_n \left[\phi^{(m+n)}_x(\bar{z},\bar{y})\right]$$

Proof: For $m=1=n$, the program s^1_1 on input z recovers program x and applies it to input (y,z). Injectivity of s^1_1 possible by the Padding Lemma.

Example: There is a recursive function $f(x)$ such that $\phi_{f(x)} = 2\phi_x$.

Proof: Let $\psi(x,y) = 2\phi_x(y)$. Then fix e with $\psi(x,y) = \phi^{(2)}_e(y,x)$.
Let $f(x) = s^1_1(e,x)$. Then $\phi_{f(x)}(z) = \phi^{(1)}_{s^1_1(e,x)}(z) = \phi^{(2)}_e(z,x) = \psi(x,z) = 2\phi_x(z)$.

Example: The function

$$f(x) = \begin{cases} 0 & \text{if } x \notin K \\ \uparrow & \text{otherwise} \end{cases}$$

is not computable. However, it is $\emptyset'$ computable.

Prop: (Reduction Principle for C.E. Sets) If A, B are computably enumerable there are A_1, B_1 computably enumerable with $A_1 \subseteq A$, $B_1 \subseteq B$, $A_1 \cap B_1 = \emptyset$, and $A_1 \cup B_1 = A \cup B$.

Defn: A sequence of computably enumerable sets $\{V_n\}_{n\in\omega}$ is uniformly computably enumerable if there is a computable function f such that $V_n = W_{f(n)}$ for all n.
Uniformly computable if there is a computable function $g(x,n)$ such that $g(\cdot,n)$ is the characteristic function of V_n for all n.

DEFN: A partial function ψ is Turing computable in A if

A set B is Turing computable in A if $\chi_B \leq_T A$; and c.e. in A iff $B = W_e^A$.

THM: (Use Principle) That $\{e\}^A(x) = y$ implies $(\exists s)(\exists \sigma \subset A)\left[\{e\}_s^\sigma(x) = y\right]$

$\{e\}_s^\sigma(x) = y$ implies $(\forall t \geq s)(\forall \tau \supseteq \sigma)\left[\{e\}_t^\tau(x) = y\right]$

$\{e\}^\sigma(x) = y$ implies $(\forall A \supset \sigma)\left[\{e\}^A(x) = y\right]$.

DEFN: The Turing degree of A is the equivalence class $\deg(A) = \{B : B \equiv_T A\}$.

PROP: That $\deg(A \oplus B)$ is the least upper bound for $\deg(A)$ and $\deg(B)$.

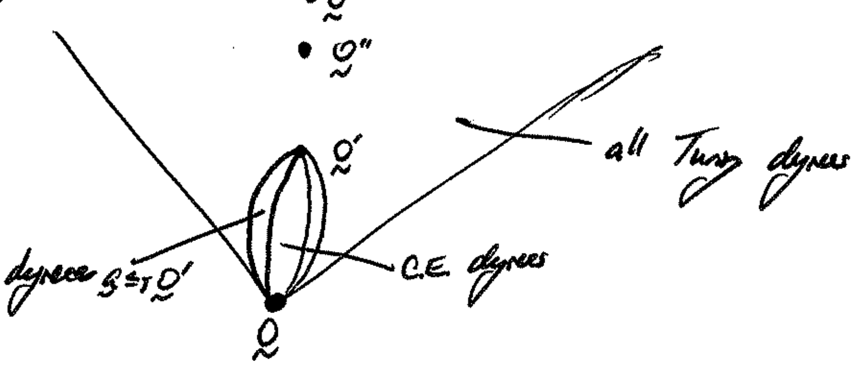
DEFN: The jump K^A of A, denoted A'. More generally, the notation $A^{(n)}$.

THM: (Jump Theorem) That A' is computably enumerable in A.

That $A' \not\leq_T A$. That A is computably enumerable in C if A is computably enumerable in B and $B \leq_T C$.

That $A' \equiv_T B'$ if $A \equiv_T B$.

REMARK: Turing Degrees $(\mathcal{D}, \leq)$ viewed as



DEFN : A set $A \subseteq \omega$ is (in) Σ^0_0 and/or Π^0_0 if A is computable.

For $n \geq 1$, a set $A \subseteq \omega$ is (in) Σ^0_n if there is a computable relation $R(x, y_1, \ldots, y_n)$ such that

$$x \in A \text{ iff } (\exists y_1)(\forall y_2) \ldots (Q y_n)\left[R(x, \bar{y})\right].$$

Is (in) Π^0_n if there is a computable relation $R(x, y_1, \ldots, y_n)$ such that

$$x \in A \text{ iff } (\forall y_1)(\exists y_2) \ldots (Q y_n)\left[R(x, \bar{y})\right].$$

Is (in) Δ^0_n if both in Σ^0_n and Π^0_n.

DEFN : Relativizations are Δ^A_n, Σ^A_n, Π^A_n.

THM : That $A \in \Sigma^0_n$ iff $\bar{A} \in \Pi^0_n$.

That $A \in \Sigma^0_n$ or $A \in \Pi^0_n$ implies $A \in \Delta^0_m$ for all $m > n$.

THM : (Post's) That $B \in \Delta^0_{n+1}$ iff $B \leq_T \emptyset^{(n)}$. That $B \in \Sigma^0_{n+1}$ iff B is c.e. in $\emptyset^{(n)}$.

DEFN : A degree $\mathbf{a} \leq_T \mathbf{0}'$ is low if $\mathbf{a}' = \mathbf{0}'$ and high if $\mathbf{a}' = \mathbf{0}''$.
A set $A \subseteq \omega$ is low/high if $\deg(A)$ is low/high.

DEFN: A modulus of convergence for $\{f_s\}_{s\in\omega}$ is a function $m(x)$ such that $f(x) = \lim_s f_s(x) = f_s(x)$ for all $s \geq m(x)$.

PROP: If $\{f_s(x)\}_{s\in\omega}$ is uniformly computable, then the least modulus is computable in any modulus.
If $f = \lim_s f_s$, then $f \leq_T m$ for any modulus m.
However $m \leq_T f$ need NOT be the case, even for the least modulus.

LEMMA: (Modulus) If A is computably enumerable and $f \leq_T A$, then there is a uniformly computable sequence $\{f_s\}_{s\in\omega}$ with $f = \lim_s f_s$ and a modulus m with $m \leq_T A$.

PROOF: Fix e and i with $A = W_i$ and $f = \{e\}^A$.
Define
$$f_s(x) = \begin{cases} \{e\}_s^{A_s}(x) & \text{if it converges} \\ 0 & \text{otherwise} \end{cases}$$
and
$$m(x) = (\mu s)(\exists z \leq s)\left[A \restriction z = A_s \restriction z \ \&\ \{e\}_s^{A_s \restriction z}(x)\downarrow\right].$$
Then m is a modulus as
$$\{e\}_s^{A_s \restriction z}(x) = \{e\}_s^{A \restriction z}(x) = \{e\}^A(x) = f(x)$$
for any $s \geq m(x)$.

LEMMA: (Limit) That $f \leq_T A'$ if and only if there is an A-computable sequence $\{f_s\}_{s\in\omega}$ such that $f = \lim_s f_s$.

PROOF: If $f \leq_T A'$, then the Modulus Lemma relativized to A assures the existence of the sequence $\{f_s\}_{s\in\omega}$.
With $f = \lim_s f_s$, define
$$A_x = \{s : (\exists t)\left[s \leq t \ \&\ f_t(x) \neq f_t(x+1)\right]\}$$
Observe A_x is finite. Also $B = \bigoplus_{x\in\omega} A_x$ is Σ_1^A, consequently $B \leq_T A'$.
Thus, given x, we can compute the least modulus $m(x) = (\mu s)\left[s \notin A_x\right]$.
Hence $f \leq_T m \oplus A \leq_T B \oplus A \leq_T A'$.

DEFN: A set A is simple if A is computably enumerable and its complement $\overline{A}$ is immune.

An infinite set is immune if it contains no infinite computably enumerable set.

THM: (Post [1944]) There exists a simple set S.

PROOF: Let $A \subseteq \omega^2$ be the relation

$$A := \{(e,x) : x \in W_e \text{ and } x > 2e\}.$$

Note that A is Σ^0_1 and hence computably enumerable.

Let ψ be a computable selector function (e.g., $\psi(e)\downarrow \to (e,\psi(e)) \in A$).

Let $S = \text{range}(\psi)$. Then S is computably enumerable, being the range of a computable selector function. The complement $\overline{S}$ is infinite as S contains at most e elements of the set $\{0,\dots,2e\}$, namely $\psi(0),\dots,\psi(e-1)$. Finally if W_e is infinite, then $W_e \cap S \neq 0$ because would have $(e,x) \in A$ with $x > 2e$ so $\psi(e)\downarrow \in S \cap W_e$.

PROOF: Satisfy requirements

$$P_e: \text{If } W_e \text{ infinite, then } A \cap W_e \neq \emptyset.$$

$$N_e: \text{That } |\overline{A}| \geq e.$$

Note that P_e is a positive requirement in that to satisfy it we enumerate elements into A whereas N_e is a negative requirement in that to satisfy it we keep elements out of A.

At stage s, let $\overline{A}_s = \{a^s_0 < a^s_1 < \dots\}$, beginning with $\overline{A}_0 = \omega$. At stage $s+1$, find least $e \leq s$ such that

① $W_{e,s} \cap A_s = \emptyset$ ② $a^s_n \in W_{e,s}$ for some $n \geq e$

Enumerate a^s_n into A. If e fails to exist, pass to next stage.