MATH 2410 QUIZ 9 3 APRIL 2009

For the autonomous linear system $\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$ with $\mathbf{A} = \begin{pmatrix} -2 & -2 \\ -2 & 1 \end{pmatrix}$ and $\mathbf{Y} = \begin{pmatrix} x \\ y \end{pmatrix}$, compute the general solution by completing the following steps.

Problem 1: (2 points) *Compute the eigenvalues of* $\mathbf{A}$.

Solution: We find the eigenvalues of $\mathbf{A} = \begin{pmatrix} -2 & -2 \\ -2 & 1 \end{pmatrix}$ by finding the roots of the characteristic polynomial (characteristic equation)

$$\det(\mathbf{A} - \lambda\mathbf{I}) = (-2-\lambda)(1-\lambda) - (-2)(-2) = 0.$$

Algebra yields $\lambda^2 + \lambda - 6 = 0$, which factors as $(\lambda - 2)(\lambda + 3) = 0$.

The eigenvalues are thus $\lambda_1 = 2$ and $\lambda_2 = -3$.

Problem 2: (2 points) *For each eigenvalue of* $\mathbf{A}$, *find an associated eigenvector.*

Solution: In order to find an eigenvector $\mathbf{V}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ for $\lambda_1 = 2$, we must solve

$$\begin{pmatrix} -2 & -2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = 2 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}.$$

Rewriting this in terms of the components, this equation is

$$\begin{cases} -2x_1 - 2y_1 = 2x_1 \\ -2x_1 + y_1 = 2y_1 \end{cases}$$

Simplifying, we obtain

$$\begin{cases} -4x_1 - 2y_1 = 0 \\ -2x_1 - y_1 = 0. \end{cases}$$

An eigenvector $\mathbf{V}_1$ for $\lambda_1 = 2$ is thus $\mathbf{V}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

In a similar fashion, to find an eigenvector $\mathbf{V}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ for $\lambda_2 = -3$, we must solve

$$\begin{pmatrix} -2 & -2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = -3 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}.$$

Doing so yields

$$\begin{cases} x_2 - 2y_2 = 0 \\ -2x_2 + 4y_2 = 0. \end{cases}$$

An eigenvector $\mathbf{V}_2$ for $\lambda_2 = -3$ is thus $\mathbf{V}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Problem 3: (2 points) *What is the general solution to the autonomous linear system* $\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$ *with* $\mathbf{A} = \begin{pmatrix} -2 & -2 \\ -2 & 1 \end{pmatrix}$ *and* $\mathbf{Y} = \begin{pmatrix} x \\ y \end{pmatrix}$*?*

Solution: Using the information from the previous parts, we know the general solution is

$$\mathbf{Y} = \mathbf{Y}(t) = k_1\, e^{2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + k_2\, e^{-3t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} k_1 e^{2t} & + & 2k_2 e^{-3t} \\ -2k_1 e^{2t} & + & k_2 e^{-3t} \end{pmatrix}.$$

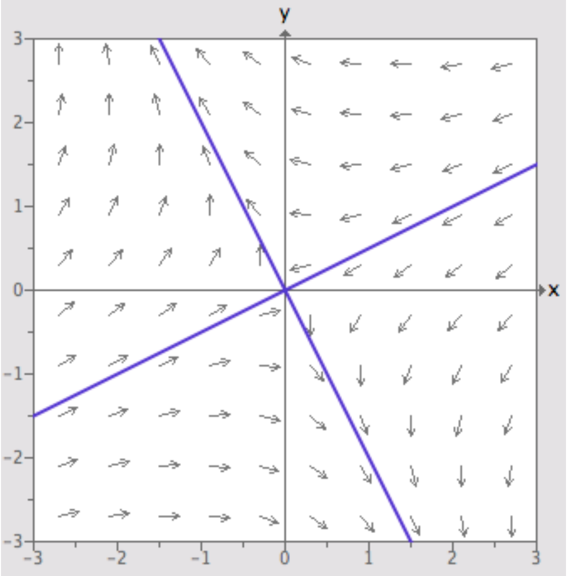
Problem 4: (2 points) *Is the equilibrium at the origin a* source*, a* sink*, a* saddle*, or* none of these*? Are there any other equilibrium to the system? How do you know?*

Solution: The equilibrium at the origin is a saddle. The reason is that λ_1 is positive (causing the associated straight-line solution to point away from the origin) while λ_2 is negative (causing the associated straight-line solution to point towards the origin).

There are no other equilibrium since $\det \mathbf{A} = (-2)(1) - (-2)(-2) = -6 \neq 0$.

Problem 5: (2 points) *Sketch the phase portrait for the system.*

Solution: The direction field and straight-line solutions are pictured



courtesy of BDH's *LinearPhasePortraits*.