Problem 1: (10 points) Consider the second-order differential equation

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} - 10y = 0.$$

Part (a): *Convert the differential equation into a system of first-order differential equations by letting* $v = \dfrac{dy}{dt}$.

Solution: With $v = \dfrac{dy}{dt}$, we have $\dfrac{dv}{dt} = \dfrac{d}{dt}\left(\dfrac{dy}{dt}\right) = \dfrac{d^2y}{dt^2} = 3\dfrac{dy}{dt} + 10y = 3v + 10y$. We thus obtain the system

$$\begin{aligned} \frac{dy}{dt} &= v \\ \frac{dv}{dt} &= 3v + 10y. \end{aligned}$$

Part (b): *Is* $\mathbf{Y}(t) = (y(t), v(t)) = (e^{3t}, 3e^{3t})$ *a solution to your system?*

Solution: We note that $\frac{dy}{dt} = \frac{d}{dt}\left(e^{3t}\right) = 3e^{3t} = v$. However $\frac{dv}{dt} = \frac{d}{dt}\left(3e^{3t}\right) = 9e^{3t}$ and $3v + 10y = 3\left(3e^{3t}\right) + 10\left(e^{3t}\right) = 19e^{3t}$. As $9e^{3t} \neq 19e^{3t}$, the vector function $\mathbf{Y}(t)$ is not a solution.

Part (c): *Find two non-zero solutions that are not multiples of one another to the second-order differential equation* $\dfrac{d^2y}{dt^2} - 3\dfrac{dy}{dt} - 10y = 0$.

Solution: Guessing a solution of the form $y = y(t) = e^{st}$ for some constant s, we compute $\frac{dy}{dt} = se^{st}$ and $\frac{d^2y}{dt^2} = s^2e^{st}$. Substituting, we must have

$$s^2e^{st} - 3se^{st} - 10e^{st} = \left(s^2 - 3s - 10\right)e^{st} = 0.$$

As e^{st} is non-zero for all t, the constant s must satisfy $s^2 - 3s - 10 = 0$. Factoring, we obtain $s = -2$ or $s = 5$. Two solutions are thus $y_1(t) = e^{-2t}$ and $y_2(t) = e^{5t}$.