Problem 1: (5 points) *Find the solution to*

$$\frac{dy}{dt} - \frac{2}{t}y = 4t^5$$

satisfying $y(1) = 2$.

Solution: As the equation is linear, we compute the integrating factor as

$$\mu(t) = \exp\left(\int -\frac{2}{t}\,dt\right) = e^{-2\ln t} = t^{-2}.$$

We therefore multiply both sides by t^{-2} obtaining

$$t^{-2}\frac{dy}{dt} - 2t^{-3}y = 4t^3.$$

We rewrite this as (making use of product rule)

$$\frac{d}{dt}\left[t^{-2}y\right] = 4t^3,$$

verifying it is correct by differentiating. Integrating both sides yields $t^{-2}y = t^4 + C$. Solving for y gives $y = t^6 + Ct^2$. The initial condition $y(1) = 2$ implies $C = 1$, so $y = t^6 + t^2$ is the solution to the initial value problem.

Note: The initial condition $y(0) = 2$ is nonsensical as the differential equation is not defined at $t = 0$ because of the division by zero.

Problem 2: (5 points) *Find the order four series approximation (i.e., find* a_0 *through* a_4*) to the differential equation*

$$\frac{dy}{dt} = y + 6\cos t$$

satisfying $y(0) = 0$. Hint*: Recall that* $\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} \pm \ldots$.

Solution: As usual, we let $y = a_0 + a_1t + a_2t^2 + a_3t^3 + a_4t^4 + a_5t^5 \ldots$. Differentiating yields $\frac{dy}{dt} = a_1 + 2a_2t + 3a_3t^2 + 4a_4t^3 + 5a_5t^4 + \ldots$. We therefore want

$$\begin{aligned} a_1 + 2a_2t + 3a_3t^2 &+ 4a_4t^3 + 5a_5t^4 + \ldots \\ &= \left(a_0 + a_1t + a_2t^2 + a_3t^3 + a_4t^4 + a_5t^5 \ldots\right) + 6\left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} \pm \ldots\right) \\ &= (a_0 + 6) + (a_1)t + (a_2 - 3)t^2 + (a_3)t^3 + (a_4 + 1/4)t^4 + \ldots \end{aligned}$$

The initial condition $y(0) = 0$ implies $a_0 = 0$. Matching constants yields $a_1 = a_0 + 6$, so $a_1 = 6$. Matching t's yields $2a_2 = a_1$, so $a_2 = 3$. Matching t^2's yields $3a_3 = a_2 - 3$, so $a_3 = 0$. Matching t^3's yields $4a_4 = a_3$, so $a_4 = 0$.

The order four approximation is thus $y \approx 0 + 6t + 3t^2 + 0t^3 + 0t^4$ (for small t).