Problem 1: (4 points) *Compute, using any method of your choice, the Laplace Transform of the function $f(t) = t$.*

Solution: Though the Laplace Transform can be computed directly from the definition $\mathcal{L}[f](s) = \int_0^\infty f(t)e^{-st}\,dt = \int_0^\infty te^{-st}\,dt$ (*hint*: integration by parts), it is simplest to use the formula $\mathcal{L}[dy/dt] = s\mathcal{L}[y] - y(0)$ to obtain

$$\mathcal{L}[1] = s\mathcal{L}[t] - 0$$

as $f(0) = 0$. Thus $\mathcal{L}[f] = \mathcal{L}[t] = \dfrac{\mathcal{L}[1]}{s} = \dfrac{1/s}{s} = \dfrac{1}{s^2}$.

Problem 2: (6 points) *Find, using Laplace Transforms, the solution to*

$$\frac{dy}{dt} + 4y = 18e^{2t}$$

with $y(0) = 0$.

Solution: Making use of the formulas $\mathcal{L}[dy/dt] = s\mathcal{L}[y] - y(0)$ and $\mathcal{L}[e^{at}] = \frac{1}{s-a}$ and linearity, taking the Laplace Transform of both sides and simplifying yields

$$s\mathcal{L}[y] - 0 + 4\mathcal{L}[y] = 18\frac{1}{s-2}.$$

Rearranging to solve for $\mathcal{L}[y]$, we find

$$\mathcal{L}[y] = \frac{18}{(s-2)(s+4)}.$$

Writing $\dfrac{18}{(s-2)(s+4)} = \dfrac{A}{s-2} + \dfrac{B}{s+4}$, we want $18 = A(s+4) + B(s-2)$. Hence $A = 3$ and $B = -3$ in the partial fraction decomposition. Thus

$$\mathcal{L}[y] = \frac{3}{s-2} - \frac{3}{s+4}.$$

Inverting the Laplace Transform yields

$$y = 3e^{2t} - 3e^{-4t}.$$