Problem 1: (10 points) *Determine the general solution of the second-order differential equation*

$$y'' + 7y' + 6y = e^{-6t}.$$

Solution: We solve it by finding the homogeneous solution y_h to the associated homogeneous equation, finding a particular solution y_p to the nonhomogeneous equation, and finding the general solution to the second-order differential equation given.

Step 1: Find the general solution to the associated homogeneous differential equation $y'' + 7y' + 6y = 0$.

We guess $y = e^{st}$, making $y' = se^{st}$ and $y'' = s^2e^{st}$. These produce $(s^2 + 7s + 6)e^{st} = 0$, so $(s+1)(s+6) = 0$. Therefore $s = -1$ or $s = -6$. The general solution to the homogneous equation is thus

$$y_h = k_1e^{-t} + k_2e^{-6t}.$$

Step 2: Find a particular solution to the nonhomogeneous differential equation $y'' + 7y' + 6y = e^{-6t}$.

Seeing that the forcing function is e^{-6t}, we need to guess $y = Ate^{-6t}$. We then determine the first and second derivatives of this guess, finding

$$y' = Ae^{-6t} - 6Ate^{-6t}$$

and

$$y'' = -6Ae^{-6t} - 6Ae^{-6t} + 36Ate^{-6t} = -12Ae^{-6t} + 36Ate^{-6t}.$$

Solving

$$\left(-12Ae^{-6t} + 36Ate^{-6t}\right) + 7\left(Ae^{-6t} - 6Ate^{-6t}\right) + 6\left(Ate^{-6t}\right) = e^{-6t}$$

gives $-5Ae^{-6t} = e^{-6t}$, or $A = -\frac{1}{5}$. Thus $y_p = -\frac{1}{5}te^{-6t}$.

Note that if the forcing function were not part of the homogeneous solution, we would not have added the t in our guess y.

Step 3: Find the general solution to the second-order differential equation given.

Using the Extended Linearity Principle, the general solution to the second-order differential equation is

$$y(t) = y_h + y_p = k_1e^{-t} + k_2e^{-6t} - \frac{1}{5}te^{-6t}.$$