Problem 1: (6 points) *For the linear system* $\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$ *with* $\mathbf{A} = \begin{pmatrix} 3 & -2 \\ 2 & 3 \end{pmatrix}$ *and* $\mathbf{Y} = \begin{pmatrix} x \\ y \end{pmatrix}$, *find the solution with* $\mathbf{Y}(0) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ *by completing the following steps.*

Part A: (2 points) *Compute the eigenvalue(s) of* $\mathbf{A}$ *and an associated eigenvector for an eigenvalue of* $\mathbf{A}$.

Solution: We find the eigenvalues of $\mathbf{A}$ by finding the roots of the characteristic polynomial (characteristic equation)

$$\det(\mathbf{A} - \lambda\mathbf{I}) = (3-\lambda)(3-\lambda) - (-2)(2) = 0.$$

Thus $(3-\lambda)^2 = -4$, so $3 - \lambda = \pm 2i$, and so the eigenvalues are $\lambda = 3 \pm 2i$.

In order to find an eigenvector $\mathbf{V} = \begin{pmatrix} x \\ y \end{pmatrix}$ for $\lambda = 3+2i$, we must solve $\mathbf{AV} = (3+2i)\mathbf{V}$. Rewriting this in terms of the components, this equation is equivalent to the system

$$\begin{cases} 3x - 2y = (3+2i)x \\ 2x + 3y = (3+2i)y. \end{cases}$$

Simplifying, we obtain that these are both equivalent to $x = iy$. An eigenvector $\mathbf{V}$ for $\lambda = 3 + 2i$ is thus $\mathbf{V} = \begin{pmatrix} i \\ 1 \end{pmatrix}$.

Part B: (2 points) *Find the solution to* $\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$ *with* $\mathbf{Y}(0) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

Solution: A solution to $\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$ is $\mathbf{Y}(t) = e^{(3+2i)t}\begin{pmatrix} i \\ 1 \end{pmatrix}$. We rewrite this as

$$\mathbf{Y}(t) = e^{(3+2i)t}\begin{pmatrix} i \\ 1 \end{pmatrix} = e^{3t}(\cos 2t + i\sin 2t)\begin{pmatrix} i \\ 1 \end{pmatrix} = e^{3t}\begin{pmatrix} -\sin 2t \\ \cos 2t \end{pmatrix} + ie^{3t}\begin{pmatrix} \cos 2t \\ \sin 2t \end{pmatrix}$$

Consequently, the general solution is

$$\mathbf{Y}(t) = k_1 e^{3t}\begin{pmatrix} -\sin 2t \\ \cos 2t \end{pmatrix} + k_2 e^{3t}\begin{pmatrix} \cos 2t \\ \sin 2t \end{pmatrix}.$$

Solving for the constants k_1 and k_2 using the initial condition $\mathbf{Y}(0) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ yields

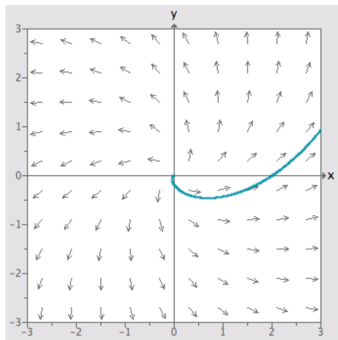
$$\mathbf{Y}(t) = 2e^{3t}\begin{pmatrix} \cos 2t \\ \sin 2t \end{pmatrix}.$$

Part C: (2 points) *Draw a sketch of the phase portrait for the system* $\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$.

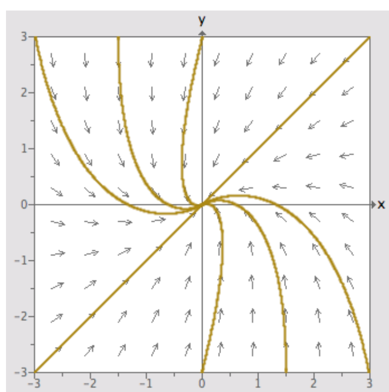
Solution: Since the eigenvalue is complex with a positive real part, the origin is a spiral source. As

$$\frac{d\mathbf{Y}}{dt}(1,0) = \begin{pmatrix} 3(1) & + & -2(0) \\ 2(1) & + & 3(0) \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix},$$

the curves must bend clockwise. We therefore sketch the following phase portrait.



Problem 2: (4 points) *Pictured is the phase portrait for a linear system* $\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$.



What can be said about the eigenvalue(s) of $\mathbf{A}$*? What can be said about the associated eigenvector(s)?*

Solution: It appears that there is a unique negative eigenvalue with eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. The presence of exactly one straight-line solution dictates the repeated eigenvalue; the origin being a sink dictates the repeated eigenvalue is negative; and the slope of the straight line solution (one) dictates an eigenvector.