Problem One (4 Points Each): True-False Questions

Indicate whether each of the following is true or false, justifying your answer.

1. *A solution to the initial value problem*

$$\frac{dy}{dt} = 3y + e^{3t} \qquad \textit{with} \qquad y(1) = e^3$$

is $y = y(t) = te^{3t}$.

TRUE. Differentiating $y(t) = te^{3t}$ using the product rule yields $y'(t) = 3te^{3t} + e^{3t}$, which we note is equal to $3y + e^{3t}$. Also $y(1) = 1 \cdot e^{3 \cdot 1} = e^3$, and so $y = y(t) = te^{3t}$ is a solution to the initial value problem.

Note that there is no need to find the general solution to the differential equation using either guess-and-check ($y = y_h + y_p$) or integrating factors.

2. *If* y_1 *and* y_2 *are solutions to the differential equation*

$$\frac{dy}{dt} = 2ty + e^{t^2},$$

then $y_1 - y_2$ *is a solution to* $\frac{dy}{dt} = 2ty$.

TRUE. Since y_1 and y_2 are solutions, we know that $\frac{dy_1}{dt} = 2ty_1 + e^{t^2}$ and $\frac{dy_2}{dt} = 2ty_2 + e^{t^2}$. Consequently

$$\frac{d}{dt}[y_1 - y_2] = \frac{d}{dt}[y_1] - \frac{d}{dt}[y_2] = \left(2ty_1 + e^{t^2}\right) - \left(2ty_2 + e^{t^2}\right) = 2ty_1 - 2ty_2 = 2t(y_1 - y_2)$$

and so $y_1 - y_2$ is a solution to the differential equation $\frac{dy}{dt} = 2ty$.

Problem Two (4 Points Each): Quick and Easy?

Answer each of the following questions. There is no need to explain your work.

1. *Give the differential equation for the logistic growth model for population with constant of proportionality k and carrying capacity N.*

With $P = P(t)$ being the population at time t, the logistic growth model with constant of proportionality k and carrying capacity N is $\dfrac{dP}{dt} = kP\left(1 - \dfrac{P}{N}\right)$.

2. *Find the general solution to the differential equation $\dfrac{dy}{dt} = ty^2$.*

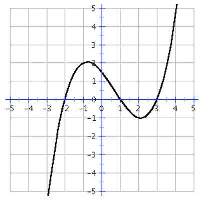
Provided $y \neq 0$, we can separate the variables and find $\dfrac{1}{y^2}\,dy = t\,dt$. We then integrate both sides, giving $-\dfrac{1}{y} = \dfrac{1}{2}t^2 + C$. We next solve for y in terms of t, obtaining $y = -\dfrac{2}{t^2 + C}$ (having replaced the constant by a new constant). The general solution is thus $y = 0$ or $y = -\dfrac{2}{t^2 + C}$ for some constant C.

3. *Draw the phase line for the differential equation $\dfrac{dP}{dt} = (P + 1)(P - 1)(P - 5)$.*

The phase line should denote equilibrium values at $P = -1$, $P = 1$, and $P = 5$. It should illustrate that P decreases when $\dfrac{dP}{dt} < 0$, i.e., when $P < -1$ and $1 < P < 5$. It should illustrate that P increases when $\dfrac{dP}{dt} > 0$, i.e., when $-1 < P < 1$ and $P > 5$.

Problem Three (5 Points Each): All From a Graph

For the graph of $f(y)$ shown, consider the autonomous differential equation $\dfrac{dy}{dt} = f(y)$.



1. *Sketch the phase line, identifying and classifying (as sinks, sources, or nodes) the equilibrium points.*

The phase line should denote equilibrium values at $P = -2$ (source), $P = 1$ (sink), and $P = 3$ (source). It should illustrate that P is decreasing when $P < -2$ and $1 < P < 3$ and that P is increasing when $-2 < P < 1$ and $P > 3$.

2. *Sketch the graphs of the solutions* $y = y(t)$ *that satisfy the initial condition* $y(0) = -3$ *and* $y(0) = 1$.

The solution with $y(0) = -3$ should be decreasing with $y(t) \to -\infty$ as $t \to \infty$. It should also have a horizontal assymptote as $t \to -\infty$ of $y = -2$. Finally, of course, it should cross the y-axis at $y = -3$.

The solution with $y(0) = 1$ should be $y(t) = 1$ for all t.

3. *If Asher claimed the existence of a solution with* $y(0) = 0$ *and* $y(2) = 2$*, would you believe him? Why or why not?*

NO. Any such solution would cross the equilibrium solution at $y = 1$. As $f(y)$ appears *nice*, the Uniqueness Theorem assures that the only solution passing through $y = 1$ is the equilibrium solution.

4. *For the one parameter family of differential equations*

$$\frac{dy}{dt} = f_\alpha(y) = f(y) + \alpha,$$

determine the bifurcation values. Explain why the system bifurcates at each of these values.

The system bifurcates at $\alpha = -2$ and $\alpha = 1$ as a consequence of there being three equilibria if $-2 < \alpha < 1$, two equilibria if $\alpha = -2$ or $\alpha = 1$, and one equilibria if $\alpha < -2$ or $\alpha > 1$.

Problem Four (10 Points Each): Slope Fields and Euler's Method

Consider the nonlinear differential equation

$$\frac{dy}{dt} = ty^3 + t.$$

1. *Draw the minitangents for the slope field at the five points* $(t, y) = (0, 0)$, $(t, y) = (-1, -1)$, $(t, y) = (-1, 1)$, $(t, y) = (1, -1)$, *and* $(t, y) = (1, 1)$.

Noting that $\frac{dy}{dt}(0, 0) = 0$, the minitangent at $(0, 0)$ will be horizontal. Similarly, the minitangents at $(-1, -1)$ and $(1, -1)$ are horizontal as $\frac{dy}{dt}(-1, -1) = 0 = \frac{dy}{dt}(1, -1)$. Noting that $\frac{dy}{dt}(-1, 1) = -2$, the minitangent at $(-1, 1)$ has slope -2; noting that $\frac{dy}{dt}(1, 1) = -2$, the minitangent at $(1, 1)$ has slope 2.

Note that, contrary to the directions, we have omitted an illustration of the slope field.

2. *Estimate both* $y(1)$ *and* $y(-1)$ *using Euler's Method with* $\Delta t = .5$ *if* $y(0) = 1$.

Euler's Method centers on the equation $y_{new} = y_{old} + \dfrac{dy}{dt} \cdot \Delta t$. When $t = 0$ and $y = 1$, we have $\frac{dy}{dt} = 0 \cdot 1^3 + 0 = 0$. Thus we estimate

$$y(.5) \approx y(0) + \frac{dy}{dt}(0, 1) \cdot \Delta t = 1 + 0\left(\frac{1}{2}\right) = 1.$$

When $t = .5$ and $y = 1$, we have $\frac{dy}{dt} = .5 \cdot 1^3 + .5 = 1$. Thus we estimate

$$y(1) \approx y(.5) + \frac{dy}{dt}(.5, 1) \cdot \Delta t \approx 1 + 1\left(\frac{1}{2}\right) = 1.5.$$

Similar computations can be done (with $\Delta t = -.5$) to estimate $y(-.5) \approx 1$ and $y(-1) \approx 1.5$.

The results can be summarized in the following table.

t	y	$\frac{dy}{dt}$
-1	1.5	
-.5	1	-1
0	1	0
.5	1	1
1	1.5	

Problem Five (10 Points Each):

Consider the initial value problem given by $\dfrac{dy}{dt} = 2y + 2$ with $y(0) = 0$.

1. *Find a (explicit) solution $y = y(t)$ for the initial value problem.*

Noting that $y(0) = 0$ assures $y \neq -1$, separating the variables yields $\dfrac{1}{y+1}\,dy = 2\,dt$. Hence $\ln|y+1| = 2t + C$. Hence $y = Ae^{2t} - 1$ for some constant A.

Alternately, rewrite it as $\frac{dy}{dt} - 2y = 2$ and compute the integrating factor $\mu(t) = \exp(\int -2\,dt) = e^{-2t}$. Multiplying both sides by it yields $e^{-2t}\frac{dy}{dt} - 2e^{-2t} = 2e^{-2t}$, or equivalently $\frac{d}{dt}\left[e^{-2t}y\right] = 2e^{-2t}$. Hence $e^{-2t}y = -e^{-2t} + C$, or $y = -1 + Ce^{-2t}$.

Alternately, note that $y_p = -1$ is an equilibrium solution (and thus a particular solution) and that $y_h = Ae^{2t}$ is a solution to the homogeneous equation. Thus $y = y_h + y_p = Ae^{2t} - 1$ is the general solution.

Regardless, putting $y = 0$ when $t = 0$ yields that the constant is one. Thus $y = e^{2t} - 1$ is a solution to the initial value problem.

2. *Find, by substituting a power series for y, the terms of order up to t^4, i.e., find the coefficients a_0, a_1, a_2, a_3, and a_4 in a Maclaurin Series for $y = y(t)$.*

We try the power series $y = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + \dots$, in which case $\frac{dy}{dt} = a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + \dots$. If this is to be a solution, we need

$$\left(a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + \dots\right) = 2\left(a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + \dots\right) + 2.$$

Matching coefficients, we need $a_1 = 2a_0 + 2$, $2a_2 = 2a_1$, $3a_3 = 2a_2$, $4a_4 = 2a_3$, and so on. The initial condition $y(0) = 0$ implies $a_0 = 0$. Consequently $a_1 = 2$, making $a_2 = 2$ and $a_3 = \frac{4}{3}$ and $a_4 = \frac{2}{3}$.

More generally, $a_n = 2^n/n!$ for $n > 0$.

Problem Six (20 Points): Pot Pouri

Elmer Ph.D. and Minnie M.S. are investigating solutions to the differential equation

$$\frac{dy}{dt} = \frac{y}{t} + 2$$

with $y(1) = 2$.

Elmer, having agonized over the Existence Theorem, suggests that there must be a solution to the initial value problem. Unfortunately, Elmer didn't review any of the techniques other than Separation of Variables (which doesn't work here), so he is at a loss on how to solve it.

Minnie, noting the division by zero if $t = 0$, *doesn't trust Elmer's appeal to the Existence Theorem. Instead, she thinks a solution might be found by using* $y = \alpha\, t \ln t$ *as a guess for the particular solution.*

1. *Explain why Elmer is correct or incorrect in thinking the Existence Theorem applies. Be as specific and precise as possible.*

2. *Determine whether Minnie's inspiration is helpful by attempting to find a solution of the form* $y = \alpha\, t \ln t$ *to* $\dfrac{dy}{dt} = \dfrac{y}{t} + 2$.

3. *Find the general solution to the differential equation* $\dfrac{dy}{dt} = \dfrac{y}{t} + 2$ *using any method of your choice.*

4. *Find the solution to the initial value problem.*

1. Elmer is correct as the differential equation is *nice* on a rectangle around $y(1) = 2$, in particular, the region $\{(t, y) : 0 < t < 2, -18 < y < 18\}$ is such a rectangle. As a direct consequence of the Existence Theorem, there is an $\varepsilon > 0$ and a solution $y(t)$ to the initial value problem defined for all t with $1 - \varepsilon < t < 1 + \varepsilon$.

2. Minnie's inspiration is also helpful. With $y_p = \alpha\, t \ln t$, we have $\dfrac{dy_p}{dt} = \alpha \ln t + \alpha$ by Product Rule. If this is to be a solution, we need $\dfrac{dy_p}{dt} = \dfrac{y_p}{t} + 2 = \dfrac{\alpha\, t \ln t}{t} + 2 = \alpha \ln t + 2$. These equate exactly when $\alpha = 2$, so $y_p = 2t \ln t$ is a particular solution.

3. Having already found a particular solution, we seek a solution to the associated homogeneous equation $\dfrac{dy}{dt} = \dfrac{y}{t}$. Separating the variables, provided $y \neq 0$ we have $\frac{1}{y}\, dy = \frac{1}{t}\, dt$; integrating yields $\ln|y| = \ln|t| + C$. Solving this for y gives $y = At$ for some arbitrary constant A (with $A = 0$ corresponding to $y = 0$).

The general solution is thus $y = y_h + y_p = At + 2t \ln t$.

3. Alternately, we can rewrite the differential equation as $\dfrac{dy}{dt} - \dfrac{y}{t} = 2$ and compute the integrating factor $\mu(t) = \exp(\int -\frac{1}{t}\,dt) = e^{-\ln t} = e^{\ln(t^{-1})} = t^{-1}$. Multiplying both sides by this, we have $t^{-1}\frac{dy}{dt} - t^{-2}y = 2t^{-1}$. Thus $\frac{d}{dt}\left[t^{-1}y\right] = 2t^{-1}$, and so $t^{-1}y = 2\ln t + C$. Multiplying by t again yields the general solution $y = 2t\ln t + Ct$.

4. We substitute $t = 1$ and $y = 2$, giving $2 = 2(1)\ln(1) + C(1)$. Hence $C = 2$. The solution to the initial value problem is thus $y = y(t) = 2t\ln t + 2t$.